Course Companion

University Mechanical Engineering

Every topic handout in one volume

exercises.lol

Contents

1. **Engineering Mechanics — Statics** 2
2. **Topic 2** 15
3. **Topic 3** 26
4. **Topic 4** 37
5. **Topic 5** 47
6. **Topic 6** 55

Engineering Mechanics – Statics

University Mechanical Engineering

What this course is, and how to use it

This is the first course in University Mechanical Engineering: **engineering statics** – the mechanics of bodies in equilibrium. Everything a machine, bridge or building does begins here, and the course's single central tool is the **free-body diagram** .

- Units 1.1 and 1.2 build the vocabulary: vectors, forces, and the discipline of isolating a body before computing anything.
- Units 1.3 to 1.5 solve equilibrium: particles first, then moments and supports for rigid bodies.
- Units 1.6 to 1.8 apply the same three equations to structures, friction and distributed loads.

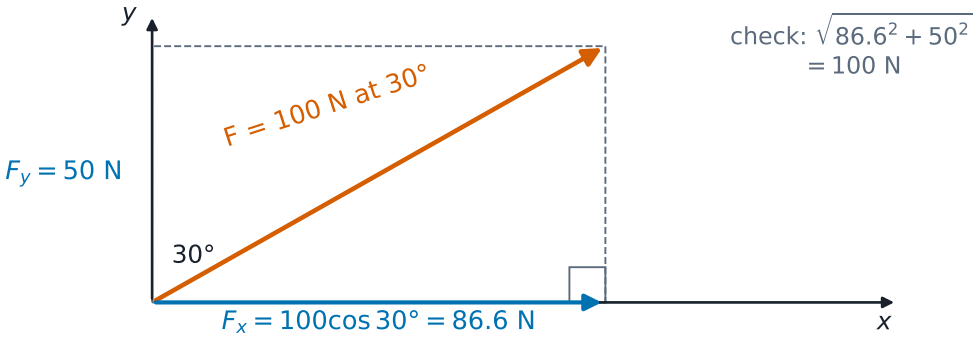
How to read each unit: a short story says why it matters; a picture comes before any formula; then the precise definitions; then worked examples that get harder, each with a check computed a second way; then the mistakes people actually make; then a three-line summary. The habit that decides success: draw the complete free-body diagram first, with every force and every reaction, and audit it before writing equations. A missing reaction is invisible in the algebra – it simply produces a confident wrong answer.

Vectors and forces

Two ropes pull a crate: one east at 30 N, one north at 40 N. Which single pull would replace them? Answering that – combining and splitting pushes – is what unit 1.1 makes precise.

A **force** is a push or pull with a **magnitude** , a **direction** and a **line of action** . Forces are vectors: they add tip-to-tail, and they resolve into components along any convenient axes. The resolution equations, with θ measured from the x-axis:

$$F_x = F\cos\theta \qquad F_y = F\sin\theta$$



Components are the sides of the rectangle; the force is its diagonal

The **resultant** of several forces is their vector sum: add the components, then rebuild the magnitude $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$ and the direction $\theta_R = \tan^{-1}(\Sigma F_y / \Sigma F_x)$. A force can be **moved along its line of action** without changing its effect on a rigid body – but never sideways off that line. Forces in SI are measured in **newtons** (N); 1 N accelerates 1 kg at 1 m/s^2. Weights are forces: $W = mg$, with $g = 9.81$ m/s^2, so a 50 kg block weighs 490.5 N.

> **Worked example 1 (resolve and check).** A 100 N force acts at 30° above the horizontal: $F_x = 100\cos 30° = 86.6$ N and $F_y = 100\sin 30° = 50$ N. Check: $\sqrt{86.6^2 + 50^2} = 100$ N – the components always rebuild the original vector, which is a free check in both directions.

> **Worked example 2 (three forces, escalated).** A bollard is pulled by three ropes: 200 N east, 150 N north, 100 N west. Sum the components first: $\Sigma F_x = 200 - 100 = 100$ N and $\Sigma F_y = 150$ N, so $R = \sqrt{100^2 + 150^2} = 180.3$ N at $\theta_R = \tan^{-1}(150/100) = 56.3°$ above east. The check by geometry: 100 and 150 are the legs of the rectangle, and 180.3 is its diagonal – the same picture as example 1, run backwards. This components-first routine replaces every tip-to-tail drawing with arithmetic that cannot tangle.

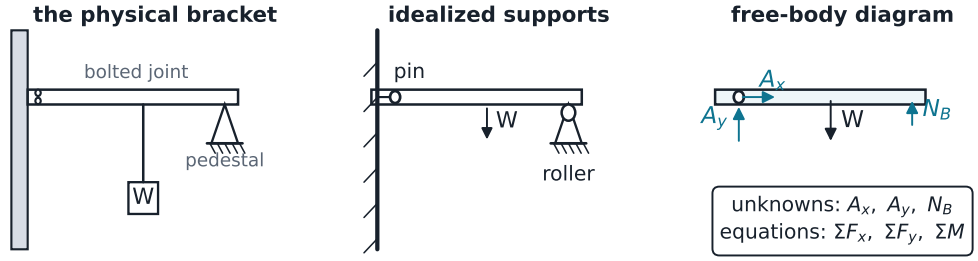
Common mistakes. Mixing the angle's reference (from the x-axis or from the force's own line) – always state which. Using cos on both components because the triangle "looks like" a cos problem: F_x takes the cosine of the angle from the x-axis, F_y the sine, every time. Reporting a magnitude without a direction – a resultant is both.

Summary: forces are vectors with magnitude, direction and line of action; resolve by cos/sin, combine by summing components, and check by rebuilding the original; move a force along its line freely, never across.

Free-body diagrams

A book rests on a table. What does the table do to the book? Nothing in the book's picture shows it – until you cut the contact and draw what it transmits. That cut, and the discipline around it, is the free-body diagram: the course's one tool every later unit uses.

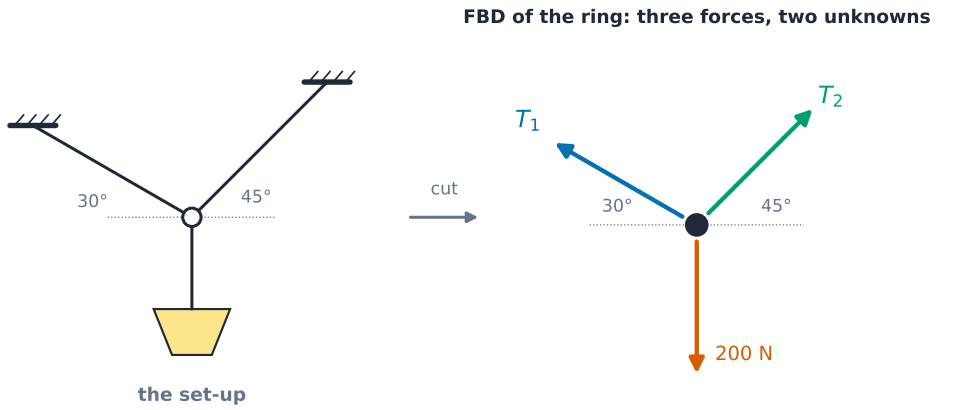
A **free-body diagram** isolates one body, draws every external force acting on it, and nothing else. It is a deliberate cut through the supports, joints and contacts, with each cut replaced by whatever it transmits.



Real object, idealized setup and free-body diagram

- **Applied loads** : gravity, cables, contacts, fluid pressure – with direction and point of application.
- **Reactions** : what each support pushes back with. The weight acts at the **centre of gravity** .
- **Internal forces never appear**: if you did not cut through the member, its force is not on this diagram.

The audit: count the unknowns, count the independent equations (two for a particle in 2D, three for a rigid body), and state whether the body is **statically determinate** – solvable by equilibrium alone.



Cut the cables and keep only what they transmit: the ring's FBD has three forces and two unknowns.

> **Worked example 1 (the lamp ring).** A lamp hangs from two cables at 30° and 45° from horizontal. The FBD of the ring is one point with three forces: the 200 N weight down and two unknown tensions along their cables. Nothing is solved yet – the diagram IS the deliverable of this step, and unit 1.3 solves it.

> **Worked example 2 (audit the beam, escalated).** A 6 m beam sits on a pin at A and a roller at B and carries one known load. The FBD shows the load, the pin's two components (A_x, A_y) and the roller's one (B_y): three unknowns against three rigid-body equations – determinate, solvable. Now audit the failure modes: miss the pin's horizontal component and the count looks friendlier but the beam is unsolved; add a second fixed end and the count rises to six against three – indeterminate, and no amount of algebra will close the gap (unit 1.5 returns to this).

Common mistakes. Drawing forces the body does not feel (the load on the *other* side of a cut). Forgetting a reaction because "the support looks rigid" – rigid is exactly why it pushes back. Drawing internal member forces on the assembly instead of cutting through them.

Summary: cut, replace, and draw only what the isolated body feels; loads have direction and point of application; the audit (unknowns vs equations) comes before any algebra.

Particle equilibrium

The lamp ring from 1.2 hangs motionless. Three forces meet there – what must be true of them? **Particle equilibrium** is the answer, and it is only two equations.

A **particle** is a body whose size does not matter – forces all act at one point.

Equilibrium in 2D is two equations:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0$$

Every force must be resolved first (unit 1.1); the equations then hold component by component. Choose axes along and perpendicular to the unknown you want to avoid: each equation then solves one unknown directly. When three forces balance, they must be **concurrent** – their lines of action pass through one point – or parallel; that geometric fact solves many problems without arithmetic.

> **Worked example 1 (the symmetric sign).** A 400 N sign hangs from two symmetric cables at 30° above horizontal. By symmetry each cable carries the same tension T; vertically, $2T\sin 30° = 400$, so $T = 400$ N. Horizontally the components cancel by symmetry – and if they did not, that would be the check that catches a blunder. Note how a steeper cable (larger angle) needs less tension: the vertical component does the holding.

> **Worked example 2 (the unsymmetric lamp, solved, escalated).** The 200 N lamp of unit 1.2: cable 1 at 30°, cable 2 at 45°. The equations, with T_1's x-component pointing left and T_2's right:
>
> $$T_1\cos 30° = T_2\cos 45° \qquad T_1\sin 30° + T_2\sin 45° = 200$$
>
> Substituting $T_1 = T_2\cos 45°/\cos 30° = 0.816\,T_2$ into the vertical equation: $T_2(0.408 + 0.707) = 200$, so $T_2 = 179.3$ N and $T_1 = 146.4$ N. Check: $146.4\sin 30° + 179.3\sin 45° = 73.2 + 126.8 = 200$ N exactly. The escalation to notice: symmetry died, the two-equation machinery did not – and the steeper 45° cable carries more, exactly as example 1 predicted.

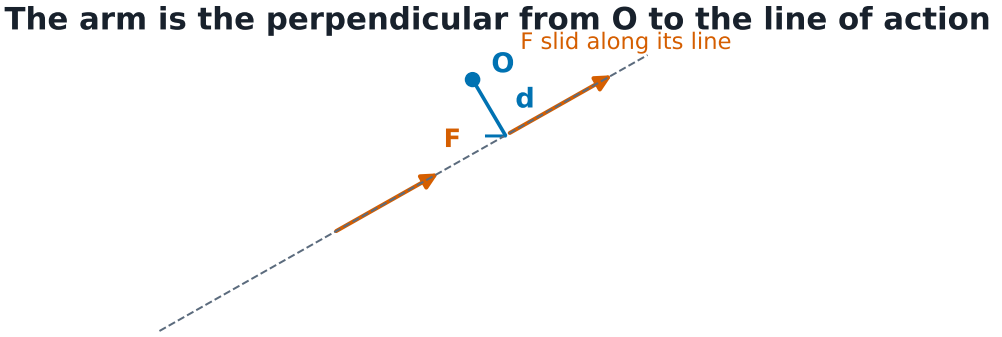
Common mistakes. Putting both x-components with the same sign because the cables "both go up" – up is the y-axis; the x-components oppose. Solving by proportion (T splits like the angles) – tension follows sines, not angle sizes. Skipping the second equation when symmetry exists: symmetry is a conclusion to verify, not to assume.

Summary: two component equations; choose axes to isolate unknowns; three forces in equilibrium are concurrent; verify symmetry instead of assuming it.

Moments and couples

Why does a longer spanner loosen a bolt that a short one cannot? The same push, a longer lever – the turning effect grows with the distance. That effect needs its own measure and its own equation.

A **moment** is the turning effect of a force about a point: $M = F \cdot d$, where d is the **moment arm** – the perpendicular distance from the point to the force's line of action. Moments are in N · m, and they carry a sign: counterclockwise positive is the usual convention.



The moment arm is the perpendicular distance – sliding the force along its line changes nothing

A **couple** is a pair of equal, opposite, parallel forces. Its moment $M = F \cdot d$ (the distance between the lines of action) is the same about **every** point – a couple has no resultant force, only a pure turning effect. The third equilibrium equation is $\Sigma M_O = 0$ about any convenient point, and the point can be chosen on an unknown force's line to eliminate it.

Deriving why a couple is reference-free. Take moments of the pair about any point O at distance a from the first force's line: $M_O = F(a + d) - F(a) = Fd$ – the a cancels. One force gains, the other loses, by exactly the same amount. That cancellation is the definition of "pure turning".

> **Worked example 1 (couple versus single force).** Two 150 N forces form a couple with lines of action 0.8 m apart: $M = 150 \times 0.8 = 120$ N · m, the same about every point in the plane. A single 240 N force with a 0.5 m moment arm also gives 120 N · m about *that* point – but only about that point; move the reference point and the moment changes. That contrast – couple constant, single-force moment reference-dependent – is the whole idea.

> **Worked example 2 (combine them, escalated).** A bracket arm feels a downward 300 N force whose line of action passes 0.4 m to the right of bolt O, plus an applied clockwise couple of 60 N · m. About O: the force gives $300 \times 0.4 = 120$ N · m clockwise, the couple 60 N · m clockwise, total 180 N · m clockwise – and about any *other* point the force part changes while the 60 N · m never does. If the bracket is in equilibrium, the bolt and its mate must supply 180 N · m of counterclockwise moment between them; the reference-independence lets you take moments about whichever bolt suits the algebra.

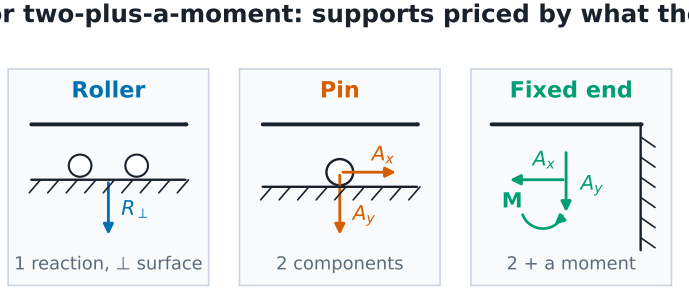
Common mistakes. Using the distance to the force's *point of application* instead of the perpendicular distance to its line of action. Dropping the couple from moment sums "because it has no force" – a couple enters EVERY moment equation, fully. Forgetting moments are signed: mixing clockwise and counterclockwise magnitudes in one sum.

Summary: $M = Fd$ about a chosen point with the perpendicular arm; couples are pure turning, the same about every point; choose the moment point to cancel unknowns.

Rigid-body equilibrium and supports

A bridge girder rests on one bolted end and one sliding end – why one bolt and not two? Because the supports' job is to supply exactly the unknowns the three equations can consume: this unit assembles forces and moments into the full rigid-body method.

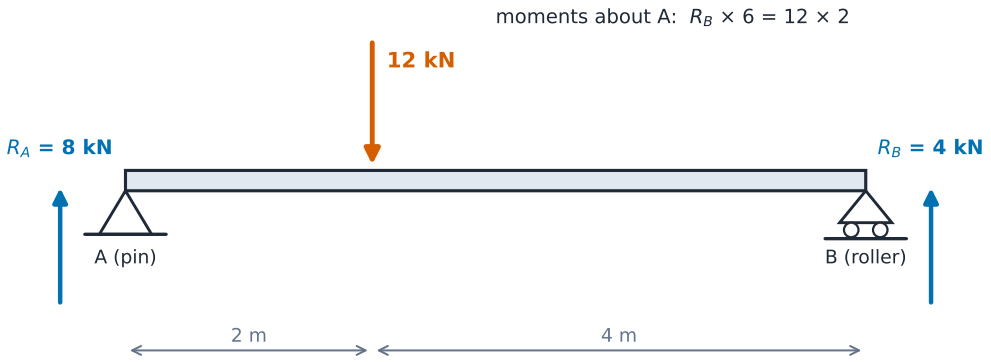
A rigid body in 2D gives three equations: $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$. Supports supply reactions, and each support type constrains a specific motion:



Roller: one reaction; pin: two; fixed end: two plus a moment

- **Roller** : one reaction, perpendicular to the surface.
- **Pin** : two reaction components, any direction.
- **Fixed support** : two force components **and** a moment.

> **Worked example 1 (the simple beam).** A 6 m beam on a pin at A and a roller at B carries a 12 kN point load 2 m from A. Moments about A: $R_B \times 6 = 12 \times 2$, so $R_B = 4$ kN; vertically $R_A = 12 - 4 = 8$ kN. Check by moments about B: $R_A \times 6 = 8 \times 6 = 48 = 12 \times 4$ ✓ – an equation that was not used to produce either answer.



Moments about A give R_B = 4 kN; the vertical sum gives R_A = 8 kN.

Worked example 2 (the overhang, escalated). The same beam, but the roller moves to 4 m from A and the 10 kN load hangs at the 6 m tip. Moments about A: $R_B \times 4 = 10 \times 6$, so $R_B = 15$ kN – larger than the load, because it holds a lever arm advantage. Vertically $R_A = 10 - 15 = -5$ kN: the pin pulls *down* by 5 kN. Check by moments about B: $R_A \times 4 + 10 \times 2 = -20 + 20 = 0$ ✓. The negative answer is not an error – it is the beam telling you the hold-down must exist, and it is why real overhangs are bolted, not just rested.

A **two-force member** carries equal, opposite forces along the line joining its two attachment points – no other loading. Spotting them turns hard structures into easy ones. Every check here recomputes an answer through an equation that was not used to produce it.

Common mistakes. Assuming every vertical reaction points up – overhangs and offset loads flip them. Taking moments about several points but using the same unknown's wrong lever arm twice: draw the distances on the FBD before summing. Calling an indeterminate assembly "solvable with more effort": 6 unknowns against 3 equations is a fact, not a challenge.

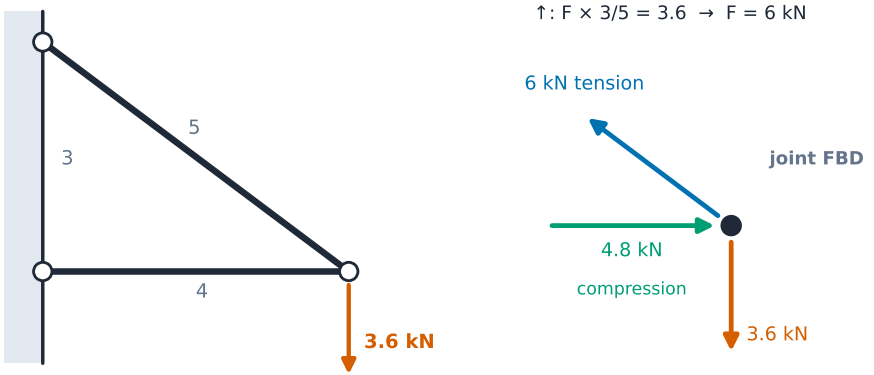
Summary: three equations, supports priced by what they prevent (1, 2, or 2+M unknowns); moment about the point that kills an unknown; negative reactions are information; two-force members are gifts.

Structures – trusses and frames

A railway bridge carries its train through hundreds of slim bars, none of which bends. The trick is the triangle: pinned triangles carry load axially, and axial means two-force means solvable – this unit is the payoff of everything before it.

A **truss** is a structure of straight two-force members joined at frictionless pins, loaded only at the joints. Members carry pure **tension** or **compression** along their axes.

Method of joints : at each joint, two force equations solve two unknown member forces. **Method of sections** : cut through three members, take one piece as a FBD, and solve three equations – ideal for one member far inside a long truss. **Zero-force members** carry zero axial force under the specified loading. At a joint with no external load or support reaction, two non-collinear members are both zero-force members. If three members meet and two are collinear, the third is a zero-force member. Symmetry is not required.



The joint's two equations: 6 kN tension in the inclined member, 4.8 kN compression in the horizontal one.

Worked example 1 (a joint at the wall). A joint at the wall is held by a horizontal member and an inclined member rising at 3-4-5 geometry, carrying a 3.6 kN load hanging down. Vertically: $F_{inclined} \times (3/5) = 3.6$, so $F_{inclined} = 6$ kN (tension). Horizontally the horizontal member takes $6 \times (4/5) = 4.8$ kN compression. Every joint of every truss is this same two-equation step.

Worked example 2 (a full truss, machine-verified, escalated). A Pratt truss of 6 m span in three 2 m panels, 2 m tall: bottom joints A–C–D–B, top joints T1 above C and T2 above D, verticals at C and D, diagonals A–T1 and T2–B, and one interior diagonal T1–D. Loads of 4 kN hang at C and D; by symmetry $R_A = R_B = 4$ kN. Solving all six joints (machine-verified, every residual zero):

- bottom chords: 4 kN tension in every panel; top chord T1–T2: 4 kN compression
- end diagonals A–T1 and T2–B: $4\sqrt{2} = 5.66$ kN compression; verticals: 4 kN tension (each one hangs its bottom joint's load from the top chord)
- the interior diagonal T1–D: **zero** – the middle panel carries no shear, because the 4 kN reaction is used up by the 4 kN load at C

This is the pattern of every simply supported truss under downward loads: it sags,

so the bottom chord stretches (tension) and the top chord shortens (compression). If your bottom chord comes out in compression, look for a sign slip.

The section method cross-checks one member without the joints: cut the right panel (bottom chord D–B and diagonal T2–B), keep the right piece (joint B) and take moments about T2, where the cut diagonal passes. With tension positive, the bottom-chord force pulls B to the left along a line 2 m below T2: $4 \times 2 - F_{bottom} \times 2 = 0$, so $F_{bottom} = 4$ kN tension – agreeing with the joint-by-joint result through a different route.

Frames and machines have multi-force members; dismember them, draw each member's FBD with the joint forces equal and opposite on the neighbour, and solve the assembly piece by piece.

Common mistakes. Loading a joint midway along a member – truss members are loaded at joints only (midway loading bends the member out of the theory). Assuming every diagonal "must" carry load: find the zero-force members first. Applying sections through four members: three cut members is the maximum three equations can buy.

Summary: joints give two equations each, sections three; zero-force members shrink the work; compression vs tension is the report; frames dismember into equal-and-opposite pairs.

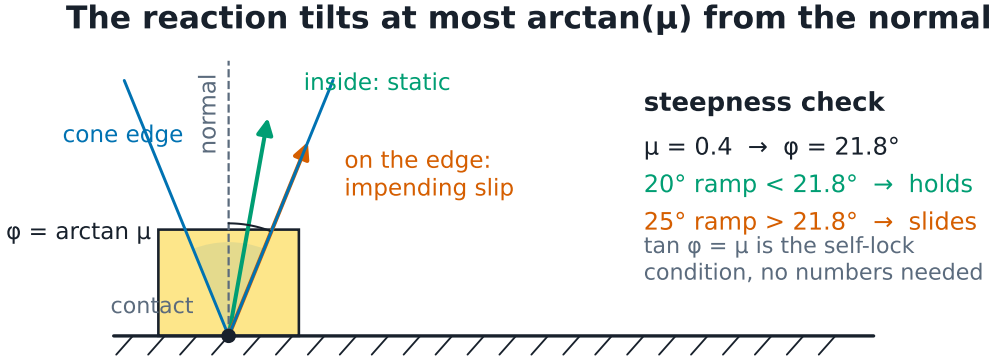
Friction

A 50 kg crate sits on a 20° ramp and does not slide. Why not – and how steep would be too steep? Friction is the one force in the course that adjusts itself, and the whole unit is learning to ask "how much does it *need*?" before computing anything.

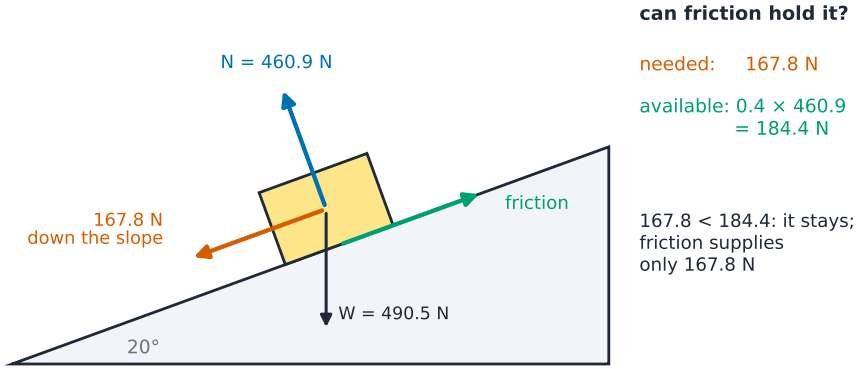
Dry **friction** opposes relative sliding at a contact. Before sliding, the friction force adjusts itself to whatever maintains equilibrium, up to a limit proportional to the normal force:

$$F \le \mu_s N \qquad F_{kinetic} = \mu_k N$$

Here the **coefficient of static friction** sets the limiting static friction, while the **coefficient of kinetic friction** applies during sliding. At the limit the contact is in **impending slip** and the full friction cone applies: the total reaction $R = \sqrt{N^2 + F^2}$ tilts at most $\phi = \tan^{-1} \mu_s$ from the normal.



The reaction tilts at most arctan(μs) from the normal – on the edge at impending slip



Friction needs 167.8 N of the 184.4 N available, so the block stays put.

Worked example 1 (the incline, and the classic comparison). A 50 kg block on a 20° incline ($\mu_s = 0.4$). Weight component down the slope: $490.5 \sin 20° = 167.8$ N; available friction: $0.4 \times 490.5 \cos 20° = 184.4$ N. Because $167.8 < 184.4$, the block stays – and the friction force actually acting is only 167.8 N, not 184.4 N. Friction takes what it needs, never what it could supply. The steepest no-slip angle follows at once: $\tan^{-1}(0.4) = 21.8°$, just above 20° – the margin is real but thin.

Worked example 2 (tip or slide, escalated). A crate 0.5 m wide and 1.0 m tall, mass 40 kg ($W = 392.4$ N), $\mu_s = 0.5$, is pulled horizontally at its top edge. Sliding needs $P_{slip} = \mu_s W = 0.5 \times 392.4 = 196.2$ N. Tipping about the front bottom edge balances the pull against the weight's moment: $P_{tip} \times 1.0 = 392.4 \times 0.25$, so $P_{tip} = 98.1$ N. The smaller one happens first: the crate **tips** at 98.1 N, well before any sliding. The design lesson: tall crates on grippy floors tip; low crates on slippery floors slide – pull low and the same crate slides instead.

Common mistakes. Setting $F = \mu N$ always: that is the *limit*; equilibrium sets the actual force, and only impending slip reaches the limit. Getting the would-be motion's direction wrong – friction opposes the motion that *would* occur, not any motion you can imagine. Comparing P values without checking which failure they belong to.

Summary: friction self-adjusts up to $\mu_s N$; compare needed against available; the cone angle $\tan^{-1}\mu_s$ is the steepest self-locking tilt; tip-versus-slide compares two pulls and takes the smaller.

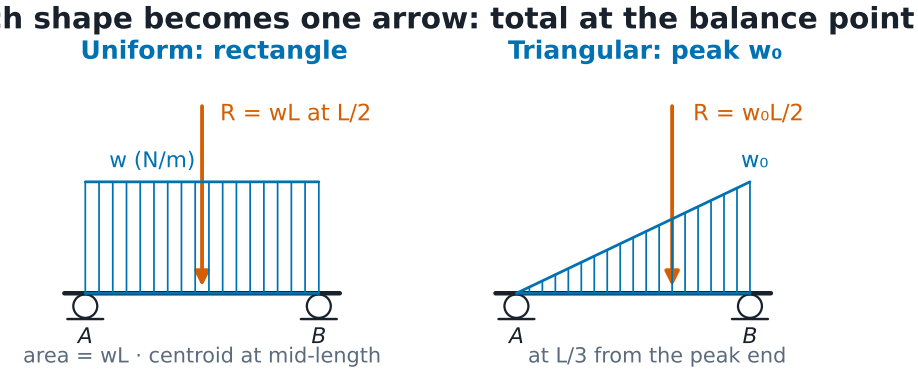
Centroids and distributed loads

Snow piles deeper at mid-span than at the eaves – the beam below does not feel a point, it feels a shape. Engineering answers with one move: replace the shape by its total at its balance point, and every earlier tool applies unchanged.

A **distributed load** spreads force over a length or area (N/m, N/m^2). A **centroid** is the load's balance point: the whole distribution can be replaced by its resultant – total magnitude acting at the centroid.

- Rectangle (uniform w over length L): $R = wL$ at the midpoint, $L/2$.
- Triangle (peak w_0 at one end over L): $R = w_0 L/2$ at $L/3$ from the peak end.

$$\Sigma M: \text{ moment of the resultant} = \text{moment of the distribution}$$



Each shape becomes one arrow: total at the balance point

Deriving the triangle's centroid. A strip at x from the zero end carries intensity proportional to x, so $\bar{x} = \int x \cdot (w_0 x/L)\,dx \,/\, \int (w_0 x/L)\,dx = 2L/3$ from the zero end, which is $L/3$ from the peak – the one-third rule, now with its origin visible.

Worked example 1 (the triangular beam). A triangular load rises from zero to 6 kN/m over 4 m on a simply supported beam. Total: $R = 6 \times 4/2 = 12$ kN acting at $4/3 = 1.333$ m from the high end. Reactions follow from two moment equations exactly as for a point load. When a uniform and a triangular load share a beam, replace each by its resultant and sum – the replacement is exact, not an approximation.

Worked example 2 (two shapes, machine-verified, escalated). A 6 m beam carries a uniform 2 kN/m over the first 3 m and a triangle rising to 4 kN/m over the last 3 m. Resultants: $R_1 = 6$ kN at 1.5 m from A; $R_2 = 6$ kN at 5 m from A. Moments about A: $R_B \times 6 = 6 \times 1.5 + 6 \times 5 = 39$, so $R_B = 6.5$ kN and $R_A = 5.5$ kN. Check about B: $5.5 \times 6 = 33 = 27 + 6$ ✓ (machine-verified).

With this the course closes its loop: loads reduce to resultants, resultants enter free-body diagrams, and the three equilibrium equations finish the job.

Common mistakes. Placing the triangular resultant at the geometric middle – it sits $L/3$ from the peak end, always. Splitting a trapezoid with the wrong peak ends. Treating the replacement as approximate: it is exact.

Summary: each shape becomes its total at its centroid – wL at $L/2$, $w_0 L/2$ at $L/3$ from the peak – then unit 1.5 finishes, exactly.

Topic 02

University Mechanical Engineering

What this course is, and how to use it

This is the second course in University Mechanical Engineering: **mathematical methods** – the mathematical toolkit the rest of the degree is written in. Course 1's statics used algebra; the rest of the degree speaks calculus, linear algebra, differential equations and statistics - taught here as instruments.

The running case is **the vibration test rig**: a mass–spring–damper rig on which the festival till's mount will be certified, plus the shafts, fields and measurements around it. Every number in the handout was computed for this course – the ODE solutions are checked by substitution, the flux by two methods, the error rates by running the methods, and the uncertainty budget by the full arithmetic.

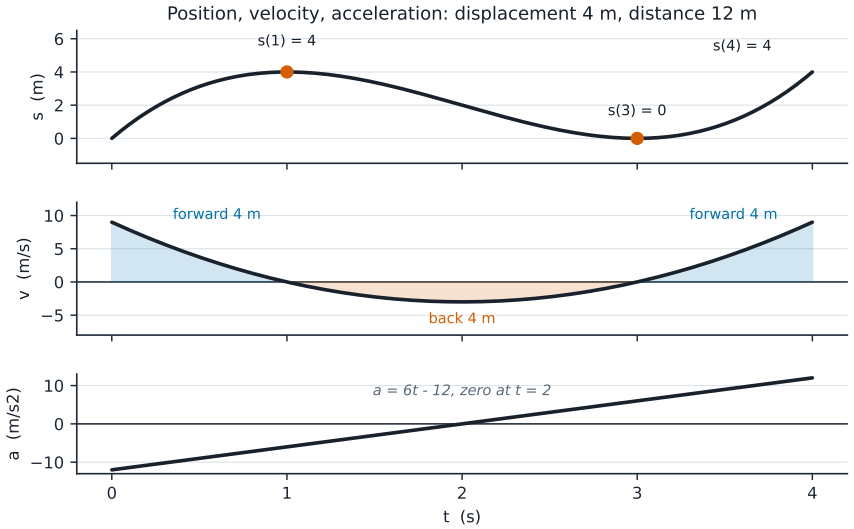
- Units 2.1–2.2 are the language; 2.3–2.4 the engine; 2.5–2.6 the world and the measurement of it.

Each unit: the question, the machinery, two escalating worked examples checked a second way, the common mistakes, a summary.

Calculus for engineering

Why it matters. The rig's trolley moves on a programmed path, and the certification report needs displacement, distance, velocity and acceleration from one position trace. Mixing them up certifies the wrong thing.

The idea, with a picture. The trolley's position: $s(t) = t^3 - 6t^2 + 9t$ metres. Velocity $v = 3t^2 - 12t + 9$ vanishes at $t = 1$ and $t = 3$ ✓; acceleration $a = 6t - 12$ is zero at $t = 2$ ✓ – the graph is a dictionary, not a decoration.



*Displacement reads the endpoints; distance reads the area - and the sign of v decides which

The definitions. Derivative : $v = ds/dt$, $a = dv/dt$; **displacement** $= \int_a^b v\,dt = s(b) - s(a)$; **definite integral** : the signed area those limits carry; **distance** $= \int_a^b |v|\,dt$. The **dimensional check**: every term of an equation carries units – $[t^3] =$ m makes $[3t^2] =$ m/s, and a term that disagrees is an error, not a style choice. **Volume of revolution** : $\pi \int y^2\,dx$ about the x-axis.

The derivation. The trolley, 0 to 4 seconds: $s(0) = 0$, $s(1) = 4$, $s(3) = 0$, $s(4) = 4$ ✓. Displacement $= s(4) - s(0) =$ **4** m ✓. Distance: forward 4 (to $t = 1$), back 4 (to $t = 3$), forward 4 again – **12** m ✓. The two numbers answer different questions, and the certification form asks for both. The fillet the mount blends into: $y = x(4-x)$ revolved about the x-axis from 0 to 2 m: $V = \pi \int_0^2 (16x^2 - 8x^3 + x^4)dx = \pi(\frac{128}{3} - 32 + \frac{32}{5}) = \pi \times 17.067 =$ **53.62** m^3 ✓ – analytic integral and 400,001-point numeric quadrature agree to four decimals.

> **Worked example 1 – the area is the story.** A junior engineer reports "the trolley travelled 4 m" from $s(4) - s(0)$. The velocity graph convicts them in one look: $v < 0$ between 1 and 3, so the trolley retrace must be counted – distance is 12. The check that never fails: compute both, and if they differ, the motion reversed somewhere; find the zeroes of v before signing anything.

Worked example 2 – escalate to the fillet checked two ways. The 53.62 m^3 blend is computed by disks; the shell method ($2\pi \int x \cdot y\,dx$ about the y-axis for the same profile) gives the volume of a *different* solid – the discipline is knowing which axis the integral belongs to. The second way for the same solid is numerical quadrature, run to 4-decimal agreement ✓. Every integral in this course earns its keep by being checked a second way, because the fillet's steel order is placed against this number.

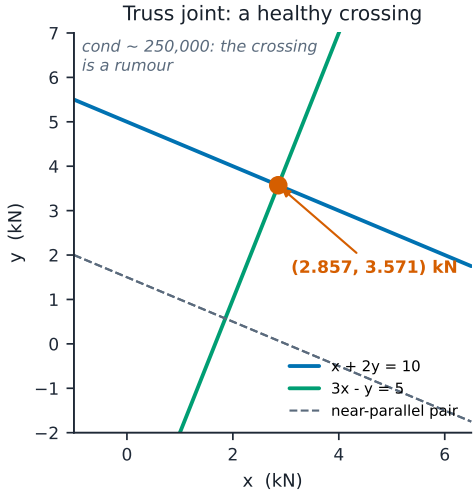
Common mistakes. Integrating $|v|$ and calling it displacement (or the reverse). Dropping units mid-derivation – the dimensional check catches algebra errors the symbols hide. And trusting an unevaluated integral on a drawing.

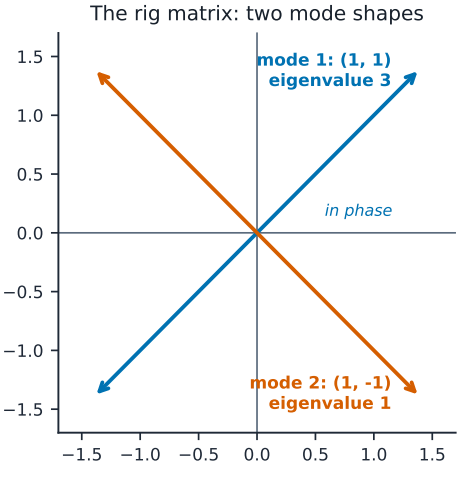
Summary. Differentiate the trace; the zeroes of v partition the story. Displacement $= s(b) - s(a)$; distance $= \int |v|$. Check dimensionally, then numerically, then sign.

Linear algebra

Why it matters. The rig's frame is a truss; the rig's vibration is a mode shape; the rig's sensor calibration is a least-squares line. Three different afternoons, one subject.

The idea, with a picture. The truss's joint equilibrium: $x + 2y = 10$, $3x - y = 5$ – two lines, one intersection $(\frac{20}{7}, \frac{25}{7}) = (2.857, 3.571)$ kN ✓. The vibration matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has **eigenvalues** 3 and 1 ✓ with mode shapes $(1,1)/\sqrt{2}$ (both masses in phase) and $(1,-1)/\sqrt{2}$ (opposed) – the two ways the rig naturally shakes.





*Where lines cross at a shallow angle, the answer is a rumour - the condition number prices it

The definitions. A **basis** : independent vectors spanning the space – any vector is one set of coefficients in each basis, and changing basis changes the coefficients, not the vector. The **condition number** $\kappa = \sigma_{max}/\sigma_{min}$ prices how the answer moves when the inputs move: errors grow by up to a factor κ. **Eigenvectors** : the directions A merely scales; the eigenvalues are the scalings.

The derivation. The conditioning exhibit, run: $B = \begin{pmatrix} 1 & 2 \\ 2 & 4.0001 \end{pmatrix}$ has $\kappa \approx$ **250,000** ✓ because its rows are nearly parallel – the two truss lines nearly coincident. Solve $B\mathbf{x} = (3.0001, 6)^T$: the answer $(7.0001, -2)$ ✓; nudge the right-hand side by 0.0001 and watch the answer move by whole units. Against this, the rig matrix has $\kappa = 3$ ✓: gentle. The rule the exhibit writes: before trusting any solved system, ask for κ – it is one line of code and it is the difference between a measurement and a guess.

Worked example 1 – modes as a basis. Any free motion of the two-mass rig is a mix of the two modes: initial displacement $(1, 0)$ is $\frac{1}{\sqrt{2}}[(1,1) + (1,-1)]/\sqrt{2}$ – half in-phase, half opposed, and each part then oscillates at its own frequency ($\sqrt{3k/m}$ and $\sqrt{k/m}$). This is why the rig's certification taps the mass at the mode frequencies: energy in decays by one mechanism at a time.

Worked example 2 – escalate to the calibration line. Ten load-cell readings against reference masses give a nearly linear cloud; least squares fits a line through a 10×2 system – which is linear algebra again, and its condition number says how much the slope can wobble from noise. The trap the exhibit generalises: the two calibration points that anchor the line are the "nearly parallel lines" of this problem – spread them apart (widen the mass range) and κ falls, with no new physics bought.

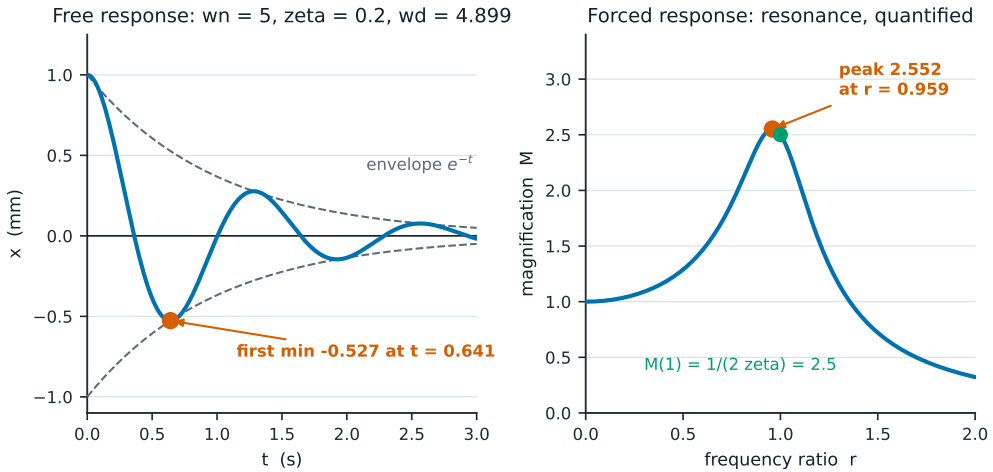
Common mistakes. Solving without looking at κ (the shallow-intersection trap). Reading eigenvectors as sacred – any nonzero multiple is the same direction. And treating basis change as a coordinate gimmick: it is the reason modes decouple.

Summary. Intersections, eigenvalues, least squares – one subject. κ prices the trust; 250,000 means the answer is a rumour. Modes are a basis: split, then solve one oscillator at a time.

Differential equations

Why it matters. The certification rig IS a differential equation: mass, damping, stiffness, force. Reading its response – free and forced – is reading whether the till's mount survives the festival.

The idea, with a picture. The rig: $\ddot{x} + 2\dot{x} + 25x = 0$ (mass 1 kg, damping 2 N · s/m, stiffness 25 N/m). **Natural frequency** $\omega_n = \sqrt{25} = 5$ rad/s, **damping ratio** $\zeta = 2/(2 \times 5) = 0.2$ – underdamped ✓, **damped frequency** $\omega_d = \sqrt{25 - 1} = \sqrt{24} = 4.899$ ✓.



*The envelope e^-t is the damping made visible; the peak at r = 0.959 is resonance made quantitative

The definitions. The **characteristic equation** $ms^2 + cs + k = 0$: its roots decide the personality – overdamped (two real), critically damped ($\zeta = 1$), **underdamped** (complex pair). The **free response**: the transient from initial conditions. The **forced response**: the particular solution tracking the input; **resonance**: the amplification near $r = \omega/\omega_n = 1$. **Magnification factor** $M(r) = 1/\sqrt{(1-r^2)^2 + (2\zeta r)^2}$; phase passes 90° at $r = 1$.

The derivation. Free response from $x_0 = 1$, $v_0 = 0$: $x(t) = e^{-t}(\cos\omega_d t + \frac{1}{\omega_d}\sin\omega_d t)$ – verified by substitution into the ODE (numerical residual $\sim 10^{-6}$ ✓). The first minimum lands at $t = \pi/\omega_d = 0.641$ s with $x = \mathbf{-0.527}$ ✓ – and the closed form $-e^{-\zeta\pi/\sqrt{1-\zeta^2}} = -e^{-0.641}$ gives -0.527 ✓: two routes, one number. Forced: $M(1) = \frac{1}{2\zeta} = \mathbf{2.5}$ ✓; the true peak sits at $r = \sqrt{1-2\zeta^2} = 0.959$ with $M = \mathbf{2.552}$ ✓ – amplification is largest just *below* resonance, the detail spec sheets exploit.

> **Worked example 1 – check a solution by substitution.** A supplier's report claims the rig settles "in about one second". From the envelope, at $t = 1$ the amplitude is $e^{-1} = 0.368$ of start – not settled; $t = 4.6$ gives 1%. The substitution check that makes the report auditable: evaluate x, $\dot{x}$, $\ddot{x}$ from the claimed formula and confirm $\ddot{x} + 2\dot{x} + 25x = 0$ to machine precision – one line of code, and it converts a claim into a fact.

> **Worked example 2 – escalate to the mount's worst case.** The festival floor vibrates at 4.79 rad/s ($r = 0.959$ – the peak). The till mount sees $M = 2.552$: a 1 mm floor motion becomes about 2.7 mm at the till (for a moving base the transmissibility $\sqrt{1+(2\zeta r)^2}\,M = 2.73$ sits just above M), and the mount's clearance is 2 mm – it fails. Three fixes, priced by the same curve: stiffen (ω_n up, r down – the curve's left slope), damp (ζ up – the peak falls as $1/2\zeta$), or isolate (ω_n well below the drive – the right slope, which is why sensitive equipment sits on soft mounts). One curve, three engineering strategies.

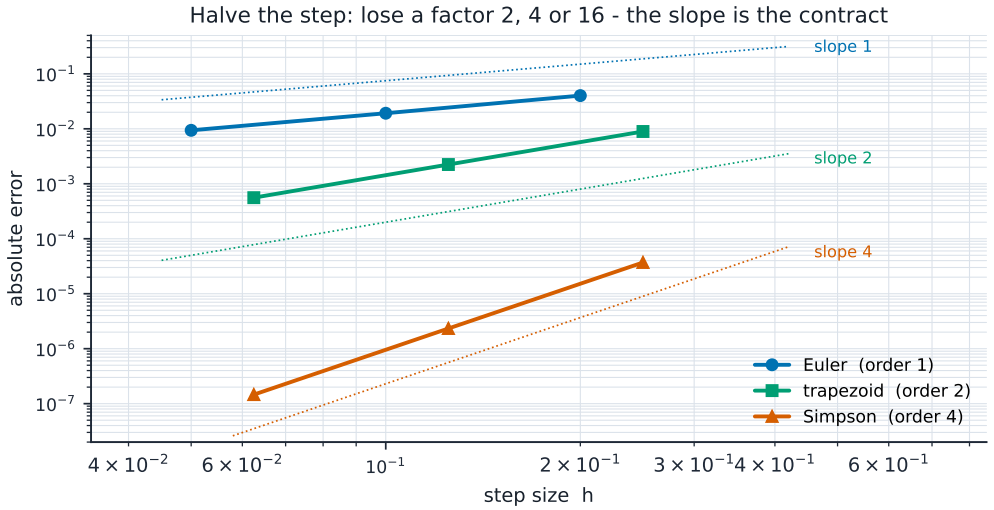
Common mistakes. Using ω_n where ω_d belongs; quoting $M(1)$ as "the peak" (it is below $r = 1$ and 2% higher); solving without a substitution check.

Summary. Roots of $ms^2 + cs + k = 0$ give the personality; $\omega_d = \omega_n\sqrt{1-\zeta^2}$. First undershoot $-e^{-\zeta\pi/\sqrt{1-\zeta^2}} = -0.527$; peak 2.552 at $r = 0.959$. Check by substitution; fix by stiffness, damping or isolation.

Numerical methods

Why it matters. The rig's real equation has a nonlinear damper and a clipped floor input – no closed form exists. Everything after "no closed form" is this unit: approximating honestly, and knowing what the approximation costs.

The idea, with a picture. Three instruments: **Euler** ($y_{n+1} = y_n + hf_n$), the **trapezoidal rule** ($\frac{h}{2}(f_0 + f_1)$ per panel), and **Simpson's rule** ($\frac{h}{3}(f_0 + 4f_1 + f_2)$ per pair) – run them, and the errors fall at their promised rates.



*The slopes are the contract: halve the step, lose a factor 2, 4 or 16 - and round-off sets the floor

The definitions. **Truncation error**: the error of the method's own

approximation, falling as h^p for order p. **Round-off error** : floating-point noise per operation, *growing* as h shrinks (more steps, more accumulation). **Total error**: the sum, a U-shaped curve whose bottom is the step size you actually want.

The derivation. The measured runs: $\dot{y} = -y$, $y(0) = 1$, Euler to $t = 1$: errors 4.0×10^{-2}, 1.9×10^{-2}, 9.4×10^{-3} at $h = 0.2, 0.1, 0.05$ ✓ – halving h halves the error (order 1). $\int_0^1 e^x\,dx = e - 1 = 1.718282$: trapezoid errors $8.9 \times 10^{-3} \to 2.2 \times 10^{-3} \to 5.6 \times 10^{-4}$ ✓ (÷4 per halving, order 2); Simpson $3.7 \times 10^{-5} \to 2.3 \times 10^{-6} \to 1.5 \times 10^{-7}$ ✓ (÷16, order 4, measured 15.9). The budget logic in one line: to buy four more digits costs about three halvings with Simpson (÷16 each), about seven with the trapezoid (÷4) and thirteen with Euler (÷2).

> **Worked example 1 – choose h with evidence.** The rig simulation must not err more than 10^{-4} over one second. Euler needs $h \approx 5 \times 10^{-4}$ (from the measured rate: error $\approx 0.19h$, so $h \approx 10^{-4}/0.19$), about 2,000 steps; the RK-class stepper this course builds toward needs $h \approx 0.1$, ten steps – two hundred times fewer, same answer. The professional sentence is not "use small steps" but "name the error budget, read h off the measured slope, and say so in the report".

> **Worked example 2 – escalate to the U-curve's floor.** Push Simpson to $n = 2^{20}$ panels and the error stops falling: each panel adds round-off $\sim 10^{-16}$, and 10^6 panels accumulate to $\sim 10^{-10}$, which dominates truncation past $n \approx 10^5$. The measured flattening IS the round-off floor – the total-error U the definitions promised, now with numbers on it. Infinite precision is not on the menu; the bottom of the U is.

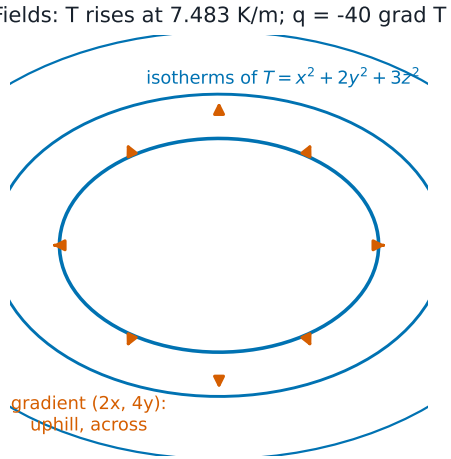
Common mistakes. Reporting a result without its error estimate. Assuming smaller h always helps (it helps until round-off wins). And validating a solver against the same physics it encodes – validate against a closed-form case first, as this unit's e^{-t} and e^x runs do.

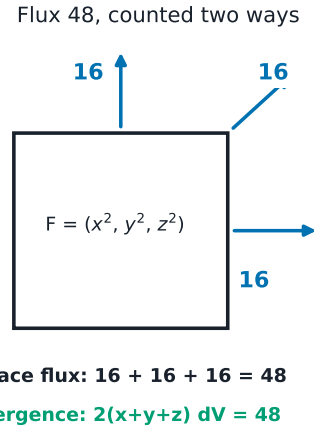
Summary. Euler O(h); trapezoid O(h^2); Simpson O(h^4) – measured, not memorised. Read h off the budget and the slope. Round-off sets the floor: the U-curve is the whole story.

Vector calculus

Why it matters. Heat leaves the rig's amplifier through a field, the cooling fan moves air through a field, and the flux/circulation theorems turn surface integrals you cannot do into volume integrals you can.

The idea, with a picture. The amplifier's temperature field $T = x^2 + 2y^2 + 3z^2$: its **gradient** $\nabla T = (2x, 4y, 6z)$ points up-hill – at $(1, 1, 1)$ that is $(2, 4, 6)$, magnitude **7.483** K/m ✓; Fourier's law turns it into heat flux $\mathbf{q} = -k\nabla T = (-80, -160, -240)$ W/m^2 for a steel chassis's $k = 40$ W/(m · K), magnitude 299.3 W/m^2 ✓.





*Gradient crosses isotherms; flux counts what leaves; the theorems let you count it the cheap way

The definitions. Gradient: steepest-ascent vector; heat flows down it. **Divergence** : $\nabla \cdot \mathbf{F}$, sources minus sinks, pointwise. **Flux** : the through-surface integral of a field. **Curl** : $\nabla \times \mathbf{F}$, the local spin. The **divergence theorem** : flux through a closed surface $= \int \nabla \cdot \mathbf{F}\,dV$. **Stokes' theorem**: circulation round a loop $= \int (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$.

The derivation. $\mathbf{F} = (x^2, y^2, z^2)$ over the cube $[0, 2]^3$: direct surface flux – the three faces at 2 each carry $\int 4\,dA = 16$, the three at 0 carry nothing – total **48** ✓. By the divergence theorem: $\nabla \cdot \mathbf{F} = 2(x + y + z)$, whose cube integral is $2 \times 3 \times 1 \times 8 = \mathbf{48}$ ✓ – two methods, one number, six surface integrals replaced by one volume integral. The spin exhibit: $\mathbf{G} = (-y, x, 0)$ has $\nabla \times \mathbf{G} = (0, 0, 2)$ and circulation $2\pi\,\mathrm{r}^2$ round a circle – 2π at $r = 1$ ✓, and Stokes says exactly that: $(0, 0, 2) \cdot (0, 0, \pi) = 2\pi$. And the field's own balance: $\nabla^2 T = 2 + 4 + 6 = 12$ ✓ – and steady conduction needs $k\nabla^2 T + \dot{q} = 0$, so at $k = 40$, $\dot{q} = -480$ W/m^3: a uniform heat *sink*, not a source ✓.

> **Worked example 1 – which integral does the theorem replace?** The fan's airflow through the rig's hexagonal grille: six awkward surface patches, or one volume integral of $\nabla \cdot \mathbf{u}$ over the box the grille encloses. The theorem is not decoration; it is the difference between an afternoon and a line – provided the surface is closed. Blow a panel open (an open surface) and the theorem declines: then you patch the hole with a convenient surface and subtract, which is how the method of images feels from inside.

Worked example 2 – escalate to the amplifier's thermal budget. The amplifier's heat leaves at $\mathbf{q} = -k\nabla T$; integrate $\dot{q} = -k\nabla^2 T = -480$ W/m^3 over its 0.0002 m^3 volume: -96 mW – the field says the block *absorbs* 96 mW, while the datasheet says the amplifier dissipates 100 mW. The sign is the story: a bowl-shaped T, coolest at the centre, describes a cooled region, and the hot amplifier's field must peak inside it ($\nabla^2 T < 0$). The size matches; the curvature does not, so this fit cannot be the amplifier's interior. This is thermal certification in miniature: field, theorem, independent number, and a sign check before any watt is believed.

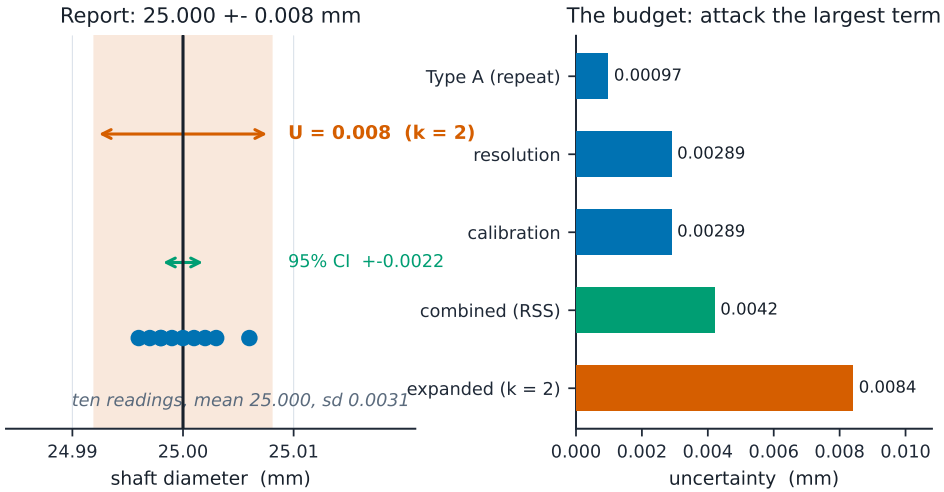
Common mistakes. Computing gradient, divergence and curl by position instead of by definition. Using the divergence theorem on open surfaces. And quoting flux without saying through what – "48" means "48 through the closed cube", nothing else.

Summary. ∇T crosses isotherms up-hill; $\mathbf{q} = -k\nabla T$ at 299 W/m^2 in the worked case. Flux 48 by surface = 48 by divergence; circulation 2π by loop = 2π by curl. Theorems trade awkward integrals for easy ones.

Probability and uncertainty

Why it matters. Every number in this course so far was exact. The rig's micrometer is not. A measurement without an **uncertainty** is a rumour with decimals, and the certification form asks for both.

The idea, with a picture. Ten micrometer readings of the rig's pivot shaft: 24.996, 25.003, 24.998, 25.001, 24.999, 24.997, 25.002, 25.000, 24.998, 25.006 mm – mean **25.000** ✓, sample sd 0.0031 mm ✓, standard error 0.00097 ✓, 95% **confidence interval** 25.000 ± 0.0022 (t = 2.262 for 9 df) ✓.



*The scatter is the data's honesty; the expanded band is the report's - and propagation carries both into every computed result

The definitions. Type A uncertainty A : from the statistics of repeated readings ($s/\sqrt{n}$). **Type B**: from calibration certificates and resolution (rectangular: $a/\sqrt{3}$; resolution: $q/\sqrt{12}$). **Combined uncertainty**: root-sum-square of independent contributions. **Expanded uncertainty** $U = ku_c$. A coverage factor $k = 2$ gives about 95% coverage when a normal approximation is appropriate and u_c is reliably estimated; small-sample estimates may require a different factor. **Propagation of uncertainty** : for $A = \pi d^2/4$, $u_A = \left|\frac{\partial A}{\partial d}\right| u_d = \frac{\pi d}{2} u_d$.

The derivation. The budget: Type A 0.00097; the 0.01 mm scale division (the third decimal is estimated by eye) gives $0.01/\sqrt{12} = 0.00289$ ✓; calibration certificate ± 0.005 gives $0.005/\sqrt{3} = 0.00289$ ✓. Combined: $\sqrt{0.00097^2 + 0.00289^2 + 0.00289^2} =$ **0.0042** mm ✓ – note the Type A term barely contributes: buying more readings would not shrink this budget. Expanded: $U = 2 \times 0.0042 =$ **0.008** mm ✓ – the report reads $\varnothing 25.000 \pm 0.008$ mm. Propagated to area: $A = \pi(25.000)^2/4 =$ **490.9** mm^2 ✓, $u_A = \frac{\pi \times 25}{2}(0.0042) = 0.165$ mm^2 ✓, reported $\mathbf{490.9 \pm 0.3}$ mm^2 ($k = 2$) ✓.

Worked example 1 – read the budget like an engineer. A new hire proposes 90 more readings. The budget says otherwise: resolution and calibration (0.00289 each) dominate the statistics (0.00097), so more readings move the fourth decimal. The instrument is the ceiling – a better micrometer (resolution 0.001 mm, certificate ± 0.002 mm) takes u_c from 0.0042 to 0.0015 mm; more patience does not. Budgets are read top-down: attack the largest term.

Worked example 2 – escalate to the rig's certification number. The mount stiffness quoted for certification is $k = mg/\delta$ measured with m known to 0.1% and δ carrying the shaft measurement's expanded U. Propagation: $\frac{u_k}{k} = \sqrt{(u_m/m)^2 + (u_\delta/\delta)^2}$ – and with $\delta = 2.50$ mm, the shaft's 0.008 mm contributes 0.32%, dominating the mass's 0.1%. The reported stiffness is $k(1 \pm 0.0034)$ – and the certification pass/fail margin must exceed that band or the test decides nothing. This is the bridge from this course to every lab that follows: no measured number leaves home without its interval.

Common mistakes. Reporting the mean's precision as the measurement's (the sd is scatter; the se is the mean's; U is the report's). Propagating uncertainties by adding them (root-sum-square for independent terms). And shrinking the easy term instead of the dominant one.

Summary. Mean 25.000, sd 0.0031, se 0.00097 – and the budget is dominated by resolution and calibration: $u_c = 0.0042$, $U = 0.008$. Propagate by RSS: 490.9 ± 0.3 mm^2. Attack the largest term; the interval travels with the number.

The project sheet (reproducible)

The rig's mathematical file, one page:

- Motion: displacement 4 m, distance 12 m over 0-4 s; fillet volume $\pi \times 17.067 = 53.62$ m^3.

- Algebra: truss joint (2.857, 3.571) kN; eigenvalues 3, 1 with modes $(1, 1)$, $(1, -1)$; condition numbers 3 and 250,008.
- ODE: $\omega_n = 5$, $\zeta = 0.2$, $\omega_d = 4.899$; first minimum -0.527; forced $M(1) = 2.5$, peak 2.552 at $r = 0.959$.
- Numerics: Euler 0.94×10^{-2} on $\dot{y} = -y$ at $h = 0.05$; trapezoid 0.56×10^{-3} and Simpson 1.5×10^{-7} on e^x at $h = 0.0625$ (ratios 2, 4, 15.9).
- Fields: $|\mathbf{q}| = 299.3$ W/m^2; cube flux 48 = 48; circulation 2π; $\nabla^2 T = 12 \to -480$ W/m^3 (a sink).
- Uncertainty: $u_c = 0.0042$, $U = 0.008$, $A = 490.9 \pm 0.3$ mm^2 – every figure recomputes from these values.

Topic 03

University Mechanical Engineering

What this course is, and how to use it

This is the third course in University Mechanical Engineering: **dynamics** – the study of motion and the forces that cause it. Course 1 (statics) drew free-body diagrams and set sums to zero; course 2 gave the mathematical instruments; this course sets those sums equal to mass times acceleration, and to their energy and momentum equivalents.

The running case continues: **the festival till** – 2.5 kg, a box 0.30 m tall and 0.20 m wide – being certified in its protective mount on the dynamics test floor. It is dropped, slid, spun, vibrated, impacted and rolled, and every number in this handout was computed for this course: the trajectories close, the Newton and energy routes give the same speed to four decimals, the impact chains conserve momentum to the digit, and the flywheel's energy ties to its spin-up work.

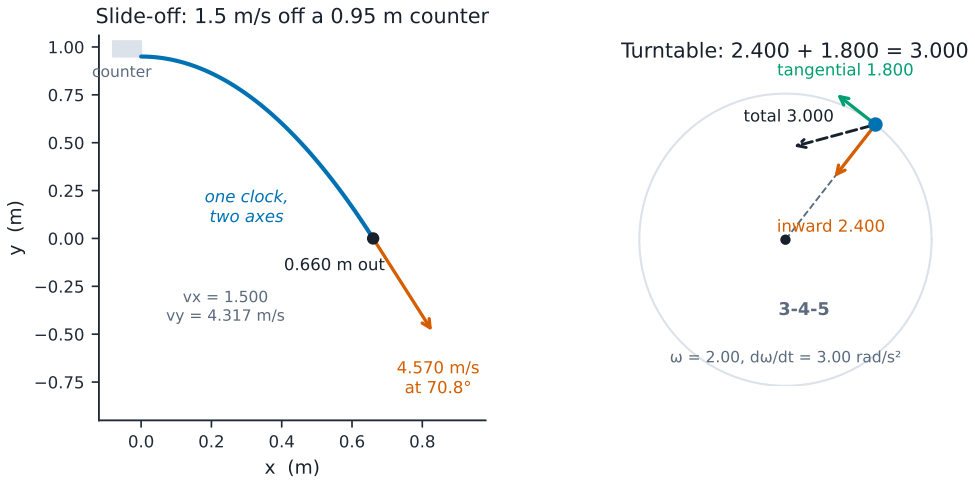
- Units 3.1–3.4 are the particle story (kinematics, forces, energy, momentum); 3.5–3.6 make the body rigid.

Each unit: the question, the machinery, two escalating worked examples checked a second way, the common mistakes, a summary.

Particle kinematics

Why it matters. Before any force can be computed, certification needs position, velocity and acceleration from the motion itself – the drop test's landing speed, the slide-off's landing point, the turntable's pull. Kinematics is reading the trace.

The idea, with a picture. The till slides off a counter 0.95 m high at 1.5 m/s horizontal: gravity owns the vertical axis, nothing owns the horizontal one, and the two share one clock. On the turntable, the same reading in polar components: inward and tangential.



*One clock, two axes - and in polar, one inward pull plus one tangential push

The definitions. A **particle** is a body whose size stops mattering – the till in free fall is one. The **position vector** $\mathbf{r}(t)$ differentiates once to **velocity** $\mathbf{v} = \dot{\mathbf{r}}$ and twice to **acceleration** $\mathbf{a} = \dot{\mathbf{v}}$; the path it traces is the **trajectory**. **Projectile motion** is the case where $a_x = 0$, $a_y = -g$: the two components share the same time t. In **polar coordinates**, with outward radial direction positive, $a_r = \ddot{r} - r\omega^2$ and $a_\theta = r\dot{\omega} + 2\dot{r}\omega$. For the fixed-radius turntable below, the inward acceleration magnitude is $r\omega^2$ and the tangential component is $r\dot{\omega}$, where **angular velocity** ω is in rad/s – never rev/s.

The derivation. The drop test, 1.80 m from rest: $v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 1.80} =$ **5.943** m/s in $t = \sqrt{2h/g} =$ **0.6058** s ✓ (the energy route of unit 3.3 will return this number). The slide-off: $t = \sqrt{2 \times 0.95/9.81} =$ **0.4401** s, range $= 1.5 \times 0.4401 =$ **0.6601** m, landing speed $\sqrt{1.5^2 + 4.317^2} =$ **4.570** m/s at $\arctan(4.317/1.5) =$ **70.8°** ✓. The turntable: $r = 0.600$ m at $\omega = 2.00$ rad/s speeding up at $\dot{\omega} = 3.00$ rad/s^2: speed $v = \omega r =$ **1.200** m/s, $a_r = \omega^2 r =$ **2.400**, $a_\theta = r\dot{\omega} =$ **1.800**, total $a = \sqrt{2.400^2 + 1.800^2} =$ **3.000** m/s^2 ✓ – a 3-4-5 triangle drawn by the physics.

> **Worked example 1 – read the landing, both numbers.** A report quotes "impact at 4.6 m/s" for the slide-off test. Which number is that, and is it right? It is $|\mathbf{v}|$ at landing: $\sqrt{v_x^2 + v_y^2}$ with $v_y = gt = 9.81 \times 0.4401 = 4.317$ gives 4.570 m/s – the report rounded up and lost the angle. The certification form wants the pair: 4.570 m/s at 70.8° below horizontal, because the mount's arrest load (unit 3.3) and its shear load split along exactly those components.

> **Worked example 2 – escalate to the spinning table.** The turntable test runs at $\omega = 2.00$ rad/s with $\dot{\omega} = 3.00$ rad/s^2. Total acceleration 3.000 m/s^2 inward-and-forward at $\arctan(1.800/2.400) = 36.9°$ ahead of the radius. The question the next unit will close: friction must supply $ma = 2.5 \times 3.000 = 7.50$ N – hold that number.

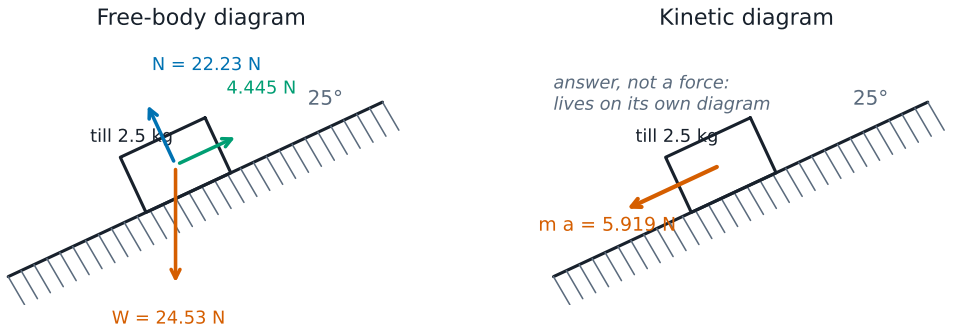
Common mistakes. Treating the two projectile components as independent clocks (there is one t); using ω in revolutions per second inside $a_r = \omega^2 r$ (convert: rev/s $\times 2\pi$); assuming speed is constant on a curve (the turntable's *speed* is constant while its *velocity* is not – direction is motion too).

Summary. Differentiate the trace; one clock for all components. Projectile: $a_y = -g$, $a_x = 0$. Polar: $a_r = \omega^2 r$ inward, $a_\theta = r\dot{\omega}$. The turntable's 2.400 + 1.800 makes 3.000 at 36.9°.

Particle kinetics

Why it matters. The mount slides or holds, slides or tips, and the certification verdict is written by Newton's second law applied to a correctly drawn body – not by intuition.

The idea, with a picture. The till on the certification ramp, 25° to the horizontal: weight, **normal force** and **friction** drawn on the free-body diagram, and the single resultant $m\mathbf{a}$ drawn beside it as the kinetic diagram.



*Forces on one diagram, mass-times-acceleration on the other - never both on the same one

The definitions. The **free-body diagram** isolates one body and draws every external force on it. The **equation of motion** is $\sum \mathbf{F} = m\mathbf{a}$: forces on one side, the kinetic diagram on the other. **Friction** is $\mu_s N$ at most while static, exactly $\mu_k N$ once sliding; the **normal force** comes from the perpendicular force equation. On the simple incline here, with no other perpendicular force or normal acceleration, it is $mg\cos\beta$.

The derivation. The ramp, $\beta = 25°$, $\mu_k = 0.20$: along the slope $mg\sin\beta - \mu_k mg\cos\beta = ma$, so $a = 9.81(0.4226 - 0.20 \times 0.9063) =$ **2.368** m/s^2; perpendicu-

lar, $N = mg\cos\beta =$ **22.23** N, friction $= \mu_k N =$ **4.445** N; from rest over 1.50 m, $v = \sqrt{2 \times 2.368 \times 1.50} =$ **2.665** m/s ✓ (unit 3.3 re-derives this exact speed from energy). The accelerating cart: the till starts to slide back when the cart's acceleration exceeds $a_{slip} = \mu_s g = 0.35 \times 9.81 =$ **3.434** m/s^2. The turntable demand: friction must give 7.50 N (unit 3.1), so $\mu \geq 7.50/(2.5 \times 9.81) =$ **0.3058**.

> **Worked example 1 – the cart test.** The shaker's transport cart brakes at 3.0 m/s^2. Does the till slide? Threshold 3.434 m/s^2 – no, with margin $(3.434 - 3.0)/3.434 =$ **12.6**%. The professional sentence names both numbers: the test pulls 3.0, the pair allows 3.43.

> **Worked example 2 – escalate to the spinning table's verdict.** The turntable needs $\mu \geq 0.3058$; the mount's measured static coefficient is 0.35. It holds – margin $(0.35 - 0.3058)/0.35 =$ **12.6**%, the same ratio because both limits scale with the normal force. The certification writes: holds at 2.00 rad/s with $\dot{\omega} = 3.00$; first slip at $\dot{\omega} = 4.09$ rad/s^2, when $r\sqrt{\omega^4 + \dot{\omega}^2} = \mu_s g$.

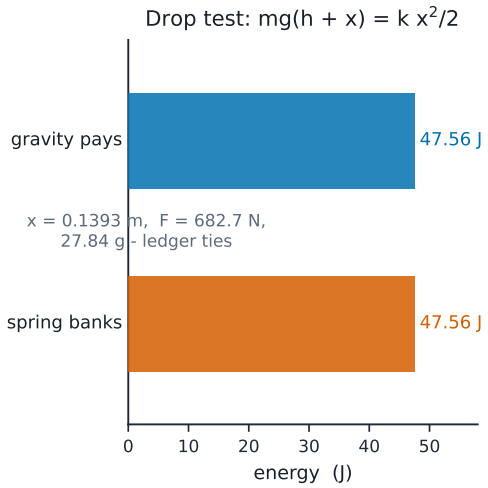
Common mistakes. Drawing $m\mathbf{a}$ as a force on the free-body diagram (it is the answer, on its own diagram); using μ_s after sliding starts; setting $N = mg$ on an incline; checking slide without checking tip (unit 3.6 closes that loop at 6.540 m/s^2).

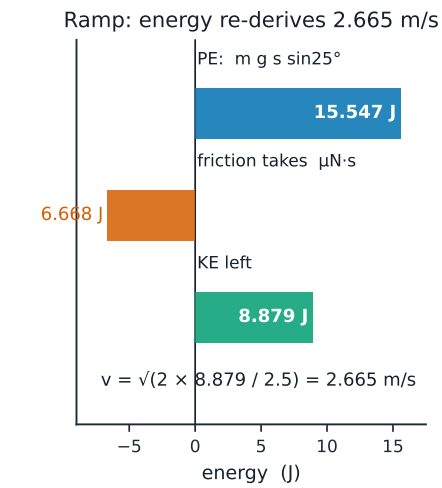
Summary. Free-body diagram, then axes along and across the motion. N from the perpendicular equation, friction μN with the right μ, then integrate. Ramp: $a = 2.368$, $v = 2.665$ m/s; cart slip 3.434; turntable needs μ 0.3058.

Work and energy

Why it matters. The mount's spring force varies with position – integrating Newton every time is possible once and pointless afterwards. Energy does the integral once and bookkeeps it forever.

The idea, with a picture. The drop test onto the mount as a spring: gravity pays in joules for the whole fall *plus* the deflection; the spring banks every joule at maximum compression.





*The ledger must balance to the last joule - that is the second-way check

The definitions. Work is $U = \int \mathbf{F} \cdot d\mathbf{r}$. Work done by an ideal spring is the decrease in its elastic potential energy; it can be positive or negative. **Kinetic energy** $T = \frac{1}{2}mv^2$ is a scalar: no components, no direction. **Potential energy** is work already banked by position: mgh for gravity, $\frac{1}{2}kx^2$ for a spring of **spring constant** k. The **energy method** is $T_1 + V_1 + U_{1\to 2} = T_2 + V_2$: one equation for how-fast and how-far questions, blind to time.

The derivation. The mount, $k = 4900$ N/m, dropped from 1.80 m: gravity supplies $mg(h+x)$ while the spring stores $\frac{1}{2}kx^2$ – the $+x$ is the deflection gravity keeps working through, the term every rushed report drops. $2450x^2 = 24.525(1.80 + x)$ gives $x_{max} =$ **0.1393** m, force $kx =$ **682.7** N, deceleration **273.1** m/s^2 = **27.84** g ✓ – and both ledger sides read **47.56** J ✓. The ramp, again: $mgs\sin\beta - \mu_k mg\cos\beta\, s = \frac{1}{2}mv^2$: $15.547 - 6.668 = 8.879$ J, $v = \sqrt{2 \times 8.879/2.5} =$ **2.665** m/s ✓ – the same speed Newton gave in unit 3.2, by a second route.

> **Worked example 1 – the arrest load without integrating.** What is the maximum force on the till? Energy: $x_{max} = 0.1393$ m, $F_{max} = kx_{max} = 682.7$ N – no force integral was ever evaluated. The average spring force is $\frac{1}{2}F_{max} = 341.4$ N, which is why energy answers and intuition checks: the peak is twice the average.

> **Worked example 2 – escalate to the rating.** The till's electronics are rated to 30 g. The drop test peaks at 27.84 g – it passes with margin $(30 - 27.84)/30 =$ **7.2**%. But halve the drop height and the peak does not halve: at $h = 0.900$ m the same equation gives $x = 0.1001$ m, $F = 490.3$ N, 20.0 g – two thirds of the rating for half the height, because gravity keeps paying through the deflection.

Common mistakes. Dropping the $+x$ in $mg(h + x)$ (the most common error in drop-test reports); double-counting gravity as both a force and a potential; friction

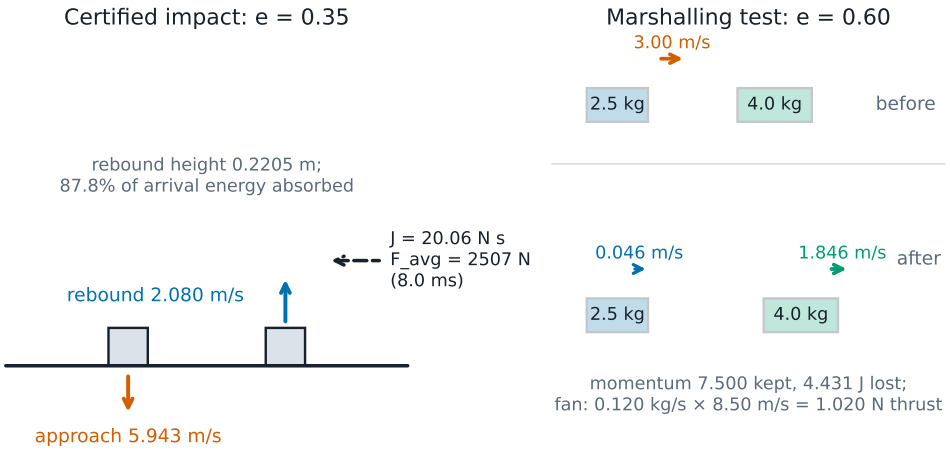
work with the wrong sign (it is $-\mu_k N s$ against the motion); asking energy for a *time* – that is unit 3.2's integration.

Summary. $T_1 + V_1 + U_{1\to 2} = T_2 + V_2$, with springs at $\frac{1}{2}kx^2$. Drop test: $x = 0.1393$ m, 682.7 N, 27.84 g, ledger 47.56 J = 47.56 J. Ramp: 15.547 − 6.668 = 8.879 J → 2.665 m/s.

Impulse and momentum

Why it matters. The certified impact lasts 8 milliseconds – too fast to integrate forces, exactly right for their time integral. Impulse and momentum are how impacts are specified, tested and survived.

The idea, with a picture. The till meets the certified floor at 5.943 m/s and leaves at 2.080 m/s; the impulse is the whole story between the two pictures.



*Two frozen pictures and one arrow of impulse between them - that is an impact analysis

The definitions. Linear momentum is $\mathbf{p} = m\mathbf{v}$ – a vector. **Impulse** is $\mathbf{J} = \int \mathbf{F}\,dt = \Delta\mathbf{p}$: forces may be unknown and wild, their time integral is not. The **coefficient of restitution** e is the separation speed over the approach speed at the contact point – 1 for elastic, 0 for dead. **Steady flow** has time-independent conditions at each fixed point. For a steady single-inlet/single-outlet stream, net external force on the fluid is $\dot{m}(\mathbf{v}_{out} - \mathbf{v}_{in})$. This includes pressure, weight and wall forces. The fluid force on a wall opposes that wall's force on the fluid.

The derivation. The drop impact at $v = 5.943$ m/s with $e = 0.35$: rebound $ev =$ **2.080** m/s, impulse $J = m(v + ev) = 2.5 \times 1.35 \times 5.943 =$ **20.06** N · s, rebound height $v_r^2/2g =$ **0.2205** m ✓. The marshalling test: the 2.5 kg cart at 3.00 m/s into a 4.0 kg cart at rest, $e = 0.60$: momentum $7.50 = 2.5v_1 + 4.0v_2$ and separation $v_2 - v_1 = 0.60 \times 3.00$ give $v_1 =$ **0.046**, $v_2 =$ **1.846** m/s ✓ – momentum checks to the digit ($2.5 \times 0.046 + 4.0 \times 1.846 = 7.50$), kinetic energy does not: 11.25 → 6.82 J, **4.431** J to dents and heat ✓. The fan: $\dot{m} = 0.120$ kg/s out at 8.50 m/s gives thrust $\dot{m}v =$ **1.020** N ✓.

> **Worked example 1 – the average force.** The certified floor contact lasts $\Delta t =$ 8.0 ms. Average force $= J/\Delta t = 20.06/0.0080 =$ **2507** N – a number the mount's welds are sized against, obtained without ever knowing the force curve. Gravity's impulse over the same 8 ms is $mg\Delta t = 0.196$ N · s, 1.0% of J: the short-impact licence to conserve momentum is now quantified, not assumed.

> **Worked example 2 – escalate to the loss budget.** How much energy may the arrest structure absorb elsewhere? The impact's kinetic budget: approach $\frac{1}{2}mv^2 =$ 44.15 J, rebound $\frac{1}{2}mv_r^2 = 5.41$ J, so the floor and mount dissipate **38.74** J – 87.8% of the incoming energy, which is what an e of 0.35 means and what the certification's crush-zone specification buys.

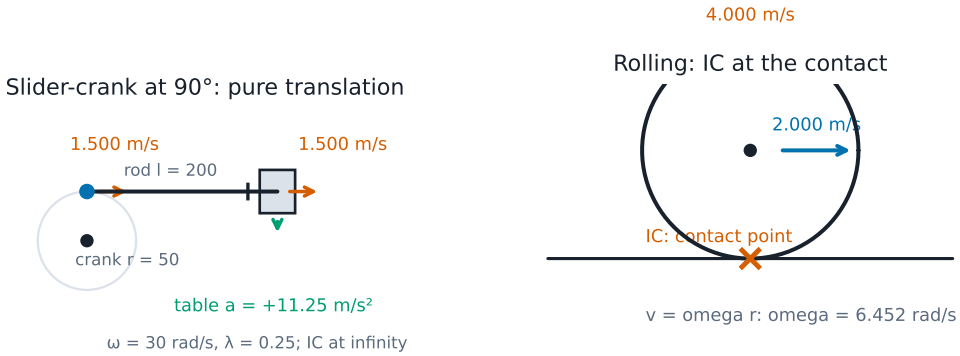
Common mistakes. Losing the vector sign (down-in and up-out momenta subtract – $J = m(v + ev)$, not $m(v - ev)$); applying e to velocities instead of the *separation/approach* pair; conserving kinetic energy because momentum is conserved (only $e = 1$ does that); drawing the system boundary so wide that external impulses sneak back in.

Summary. $\sum \mathbf{J} = \Delta\mathbf{p}$; e = separation over approach. Impact: $J = 20.06$ N · s, average force 2507 N over 8 ms. Carts: 0.046 and 1.846 m/s, 4.431 J lost. Fan thrust $\dot{m}v = 1.020$ N.

Rigid-body kinematics

Why it matters. The shaker's drive is a mechanism, not a particle: crank, rod, table, each point moving differently, joined by geometry. Rigid-body kinematics is constraint before differentiation.

The idea, with a picture. The shaker's slider-crank at the quarter-turn position, and the transport wheel rolling without slip – two machines, one method: find the constraint, then differentiate.



*One mechanism, two velocities, one zero-velocity point - the geometry does the differentiating

The definitions. A **rigid body** keeps every pair of its points at fixed distance. The **slider-crank mechanism** converts rotation to reciprocation: crank r, rod l, ratio $\lambda = r/l$. The **instant centre** of any body planar motion is the point of zero velocity at that instant – every other point turns about it *now*. **Rolling without slipping** ties the wheel to the ground: $v = \omega r$, contact point at rest; in a **rotating frame** the same motion is described with time-varying basis vectors.

The derivation. The shaker drive, $r = 50$ mm, $l = 200$ mm, $\omega = 30$ rad/s: piston position $x = r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta}$. At $\theta = 90°$: $x = \sqrt{l^2 - r^2} = \mathbf{0.1936}$ m, and differentiating, $\dot{x} = -r\omega = \mathbf{-1.500}$ m/s while $\ddot{x} = +r\omega^2\lambda = \mathbf{+11.25}$ m/s² ✓; at $\theta = 0°$ the acceleration peaks at $\ddot{x} = -r\omega^2(1+\lambda) = \mathbf{-56.25}$ m/s² ✓ – the same crank swings its table through a 5:1 acceleration range. The wheel, $r = 0.310$ m rolling at 2.00 m/s: $\omega = v/r = \mathbf{6.452}$ rad/s, contact point at rest – the **instant centre** – centre at 2.00, top at **4.000** m/s ✓.

> **Worked example 1 – the rod is translating at quarter-turn.** At $\theta = 90°$ the crank pin's velocity is horizontal (perpendicular to the vertical crank) and the piston's velocity is horizontal by its slide – both perpendiculars to the rod's direction are parallel, so the rod's instant centre is at infinity: pure translation, every rod point at 1.500 m/s ✓. The check that never fails: compute the rod's end-point speeds independently – both 1.500 m/s, same direction.

> **Worked example 2 – escalate to the fatigue spec.** The table's acceleration cycles between +11.25 at quarter-turn and −56.25 m/s² at dead centre each revolution – a 5:1 load range at 30 rad/s = 4.77 Hz. The mount's fatigue file is written from exactly this kinematic output: 2.9×10^6 cycles per week at the peak, the number the weld sizing (unit 3.6) consumes.

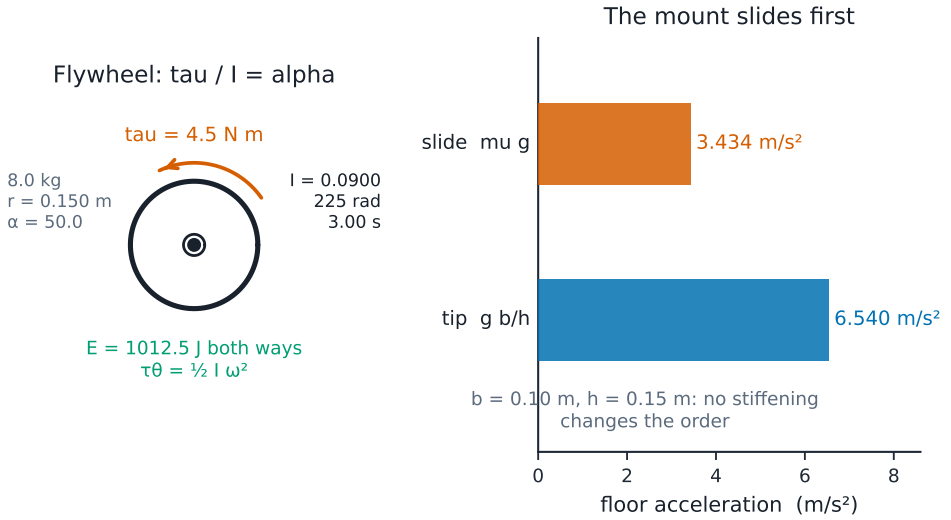
Common mistakes. Feeding rev/s into $a = \omega^2 r$; believing the instant centre is a fixed pivot (it travels with the body's configuration); forgetting the wheel's top moves at $2v$ while the contact sleeps; differentiating a constraint without stating the frame.

Summary. Constraint geometry first, differentiation second. Slider-crank at 90°: $v = r\omega = 1.500$ m/s, $a = r\omega^2\lambda = 11.25$ m/s²; extremes 5:1. Rolling: $v = \omega r$, IC at contact, top at $2v$.

Rigid-body kinetics

Why it matters. Spin stores energy and resists torque by geometry, not just mass; the certification file's last rows are the flywheel's inertia and the mount's tip-versus-slide verdict.

The idea, with a picture. The spin-test flywheel – a disk with a torque and a number I – beside the two failure modes of the mount on a rough accelerating floor: slide at 3.434, tip at 6.540 m/s².



*Mass resists push, inertia resists spin - and a box meets its smaller limit first

The definitions. The **mass moment of inertia** $I = \int r^2\,dm$ is the spin analogue of mass – a disk about its axle carries $\frac{1}{2}mr^2$. **Torque** τ turns it: for rotation about a fixed axis with constant I, $\sum\tau = I\alpha$ with **angular acceleration** α in rad/s². The axial **angular momentum** is $L = I\omega$ for this fixed-axis model. Total angular momentum about a fixed inertial point is conserved when the net external torque about that point is zero. **Tipping** begins when the resultant force crosses the base edge: for a box of half-width b and centre-of-mass height h, at $a = gb/h$.

The derivation. The flywheel, $m = 8.0$ kg, $r = 0.150$ m, $\tau = 4.5$ N · m: $I = \frac{1}{2}mr^2 =$

0.0900 kg · m^2, $\alpha = \tau/I =$ **50.00** rad/s^2, so 3.00 s of spin-up reaches $\omega =$ **150** rad/s through $\theta = \frac{1}{2}\alpha t^2 =$ **225** rad = **35.81** revolutions ✓, storing $\frac{1}{2}I\omega^2 =$ **1012.5** J ✓ – equal to the torque's work $\tau\theta = 4.5 \times 225 = 1012.5$ J ✓, the energy ledger again. The rolling disk down 15°: $a = g\sin\beta/(1 + I/mr^2) = \frac{2}{3} \times 9.81 \times 0.2588 =$ **1.693** m/s^2. The mount's verdict: tip at $a_{tip} = g \times 0.10/0.15 =$ **6.540** m/s^2, slide at $\mu_s g =$ **3.434** – it **slides first**, and no stiffening changes that order.

> **Worked example 1 – measure I by rundown.** With the drive off, the flywheel coasts from 150 rad/s to rest in 40.0 s: $\alpha = -3.75$ rad/s^2, so bearing friction is $\tau_f = I|\alpha| =$ **0.3375** N · m – the inertia measured from a stopwatch, no geometry taken on faith. The energy cross-check: friction dissipates $\frac{1}{2}I\omega^2 = 1012.5$ J over $\theta = \omega^2/2|\alpha| = 3000$ rad, and $0.3375 \times 3000 = 1012.5$ J ✓.

> **Worked example 2 – escalate to the rolling split.** A 5.0 kg disk rolls 1.50 m down the 15° ramp: $a = 1.693$ m/s^2 gives $v = \sqrt{2 \times 1.693 \times 1.50} =$ **2.253** m/s. The energy split: translational $\frac{1}{2}mv^2 =$ **12.70** J, rotational $\frac{1}{2}I\omega^2 = \frac{1}{4}mv^2 =$ **6.35** J, total **19.04** J – and the drop was $mg \times 1.50\sin 15° = 19.04$ J ✓. Two thirds translation, one third rotation: the disk's own geometry taxes every ramp.

Common mistakes. Quoting I without its axis (about the axle and about the edge differ by mr^2); using the tipping formula with full width instead of half-width, or rim height instead of centre-of-mass height; adding rolling energies without the $(1+I/mr^2)$ factor; treating $\sum\tau = I\alpha$ as valid about any point (a fixed axle or the centre of mass, or pay with extra terms).

Summary. $\sum\tau = I\alpha$, disk $I = \frac{1}{2}mr^2$: flywheel 0.0900, 50.00 rad/s^2, 1012.5 J both ways. Rolling disk $a = \frac{2}{3}g\sin\beta = 1.693$, energy split 2:1. The mount slides at 3.434 before it tips at 6.540.

The project sheet (reproducible)

The certification file's dynamics page, one page:

- Kinematics: drop from 1.80 m lands at 5.943 m/s in 0.6058 s; slide-off lands at 4.570 m/s, 70.8°, 0.6601 m out; turntable 2.400 + 1.800 = 3.000 m/s^2.
- Newton: ramp $a = 2.368$ m/s^2, $v = 2.665$ m/s, $N = 22.23$ N; cart slip at 3.434; turntable needs μ 0.3058.
- Energy: mount compresses 0.1393 m, force 682.7 N, 27.84 g, ledger 47.56 J = 47.56 J; ramp 15.547 − 6.668 = 8.879 J → 2.665 m/s.
- Momentum: impulse 20.06 N · s, average force 2507 N (8 ms), rebound 2.080 m/s; carts 0.046 + 1.846 m/s, 4.431 J lost; fan thrust 1.020 N.
- Mechanism: quarter-turn $v = 1.500$ m/s, $a = 11.25$ m/s^2, extremes 5:1; wheel $\omega = 6.452$ rad/s, top 4.000 m/s.
- Rigid body: flywheel $I = 0.0900$, $\alpha = 50.00$, 225 rad = 35.81 rev, 1012.5 J; rolling disk 1.693 m/s^2, split 12.70 + 6.35 = 19.04 J; mount slides at 3.434 before tipping at 6.540 m/s^2.

Recompute any line from $g = 9.81$, $m = 2.5$ kg and the dimensions above – a reader with a calculator can check every number, and the machine suite that generated them runs in under a second.

Topic 04

University Mechanical Engineering

What this course is, and how to use it

This is the fourth course in University Mechanical Engineering: **engineering materials** – why materials behave as they do, how that behaviour is measured, and how a material is chosen, processed and certified for a part.

The running case closes a loop opened in course 3: the till mount was a spring of $k = 4900$ N/m – this course asks what it is *made of*. Four candidates are followed throughout: **low-carbon steel** low-carbon steel, **aluminium alloy 6061**, **cast nylon 6** and a **carbon-fibre/epoxy laminate**. Every number in this handout was computed for this course – the elasticity chains recompute, the phase fractions sum to one, the probability curves integrate, and the selection index ranks the candidates.

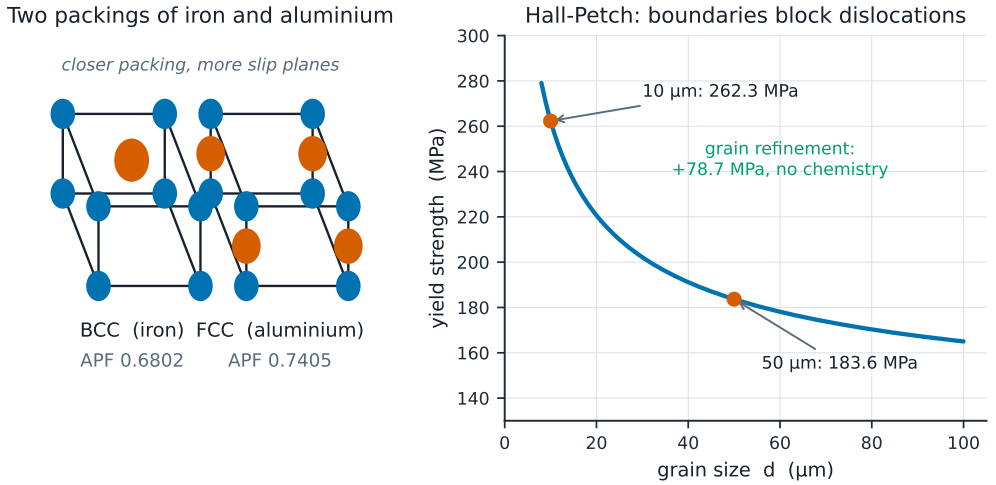
- Units 4.1–4.2 are why and how much; 4.3–4.4 the families; 4.5–4.6 the making and the surviving.

Each unit: the question, the machinery, two escalating worked examples checked a second way, the common mistakes, a summary.

Structure and bonding

Why it matters. The mount's stiffness, yield and failure all trace to atoms and microstructure. Choosing blind is choosing by rumour.

The idea, with a picture. Bonding type sets the baseline: metallic bonds give ductile conductors (steel, aluminium), covalent networks give stiff insulators (ceramics), polymer chains give light viscoelastic solids (nylon). Above the bond, the **crystal structure** packs atoms: body-centred cubic and face-centred cubic differ in density of packing – and packing plus defects sets how a metal yields.



*Denser packing, finer grains: structure writes the property

The definitions. The **atomic packing factor** is the fraction of a unit cell filled by atoms treated as hard spheres: $\pi\sqrt{3}/8 = \mathbf{0.6802}$ for BCC (iron at room temperature) and $\pi/(3\sqrt{2}) = \mathbf{0.7405}$ for FCC (aluminium). A **dislocation** is a line defect whose motion is plastic flow; a **grain boundary** is a wall between crystals that blocks dislocations – smaller grains mean more walls. **Grain refinement** strengthens by that road: the Hall-Petch relation $\sigma_y = \sigma_0 + k/\sqrt{d}$.

The derivation. The bracket steel: $\sigma_0 = 120$ MPa and $k = 0.45$ MPa $\cdot\sqrt{\text{m}}$. At $d = 50$ μm: $\sigma_y = 120 + 0.45/\sqrt{50 \times 10^{-6}} = 120 + 63.6 = \mathbf{183.6}$ MPa ✓; refine to $d = 10$ μm: $120 + 142.3 = \mathbf{262.3}$ MPa ✓ – a gain of **78.7** MPa from processing alone, no chemistry changed ✓ (the two points are the picture's right panel).

> **Worked example 1 – read the packing.** Aluminium is FCC, iron is BCC: the APF ratio is $0.7405/0.6802 = 1.089$ – aluminium's lattice fills 8.9% more of space with equally sized atoms, the geometric root of its lower density per atom and its many slip planes: FCC metals form and deform easily (aluminium foil), BCC iron needs warmth to match it.

> **Worked example 2 – escalate to the mount's bracket.** The specification demands $\sigma_y \geq 250$ MPa with 20% elongation. Hall-Petch says the grain size must satisfy $120 + 0.45/\sqrt{d} \geq 250$, so $d \leq (0.45/130)^2 = \mathbf{1.20 \times 10^{-5}}$ m $= \mathbf{12.0}$ μm ✓ – grain refinement alone reaches the target (10 µm gives 262.3), and the elongation is barely touched because boundaries block dislocations without embrittling the way carbon does.

Common mistakes. Treating APF as density (it is a geometric ratio; atomic mass still decides ρ). Quoting Hall-Petch outside its measured material and grain-size range (other mechanisms can dominate at nanometre grain sizes). Forgetting that disloca-

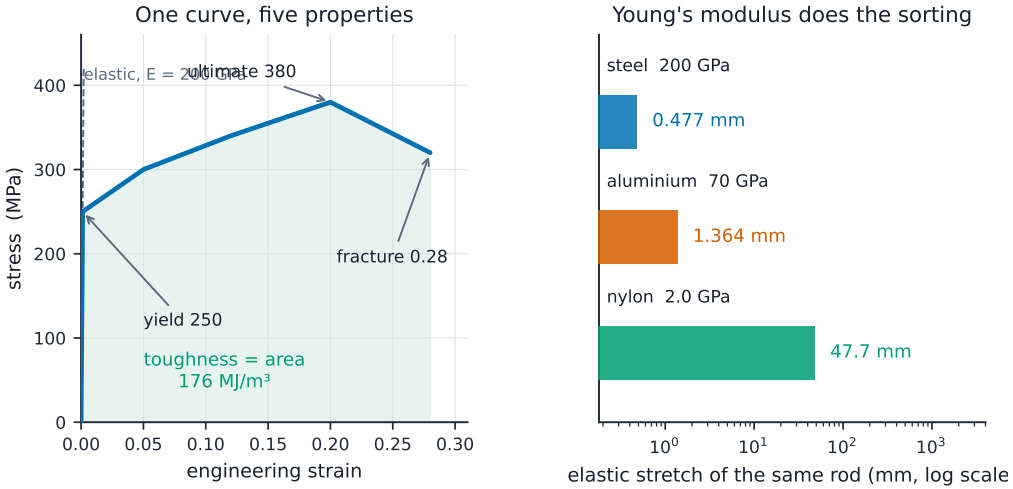
tions are the *mechanism* of ductility, not a defect to eliminate.

Summary. Bonding sets family behaviour; packing: BCC 0.6802, FCC 0.7405. Dislocations glide, boundaries block: $\sigma_y = \sigma_0 + k/\sqrt{d}$. The bracket steel: 183.6 → 262.3 MPa for 50 → 10 µm.

Stress-strain and elastic behaviour

Why it matters. Certification numbers – stiffness, yield, toughness – are read from one curve, and the test that produced it has conventions that make or break comparisons.

The idea, with a picture. The mount rod test: a 10.0 mm rod, 0.50 m long, pulled to 15.0 kN. Steel stretches elastically by less than half a millimetre; the same load in aluminium or nylon is a different story.



*One curve, five properties - and E sorts the elastic league

The definitions. Engineering stress is $\sigma = F/A_0$ on the *original* area; **true stress** uses the current area. For uniform uniaxial deformation at constant volume, before necking, $\sigma_t = \sigma(1+\varepsilon)$. For uniform extension, true strain is $\varepsilon_t = \ln(1+\varepsilon)$. **Young's modulus** E is the elastic slope: 200 GPa (steel), 70 GPa (aluminium), 2.0 GPa (nylon). The **yield strength** is where plastic flow begins (0.2% offset convention); **toughness** is the *area* under the curve – energy to fracture, in MJ/m^3; **ductility** is strain at fracture.

The derivation. The rod: $A_0 = \pi(0.010)^2/4 = 7.854 \times 10^{-5}$ m^2, $\sigma = F/A_0 = \mathbf{191.0}$ MPa ✓ – below the steel's 250 MPa yield, so purely elastic: $\varepsilon = \sigma/E = 9.55 \times 10^{-4}$, stretch $= \varepsilon L = \mathbf{0.477}$ mm ✓. Aluminium at the same load: $191.0/70\,000 = 2.73\times10^{-3}$, stretch **1.364** mm – 2.86× more, exactly the modulus ratio 200/70 ✓ (the second-way check: deflections scale inversely with E). Nylon: $191.0/2000 = 9.55 \times 10^{-2}$ → 47.7 mm – off the elastic map, which is why the mount's stops exist. True vs engineering at $\varepsilon = 0.20$: $\varepsilon_t = \ln 1.20 = \mathbf{0.1823}$, $\sigma_t = 1.20\sigma$ – the two languages diverge exactly when necking starts to matter ✓. Toughness of the bracket steel (area of the curve in the picture, trapezoid over the marked points): **176** MJ/m^3 ✓.

> **Worked example 1 – the 0.2% offset, done honestly.** A supplier's plot shows no sharp yield. Draw the elastic line, shift it 0.002 in strain; where it crosses the curve is σ_y. The certificate reads 250 MPa – but for a *different* batch the same convention gave 310: the convention, not the material, changed? No – check the gauge length. Ductility percentages only compare at equal gauge length, because necking localises: the report without gauge length is a rumour.

> **Worked example 2 – escalate to toughness budgets.** The drop test of course 3 delivers 47.56 J to the mount. A bracket volume of $47.56/176 = \mathbf{0.270}$ cm^3 of this steel could absorb the whole blow plastically – the real bracket is 4.0 cm^3, so the drop rides at 6.8% of its energy capacity ✓ (two courses tie: course 3's joules, this course's area-under-curve).

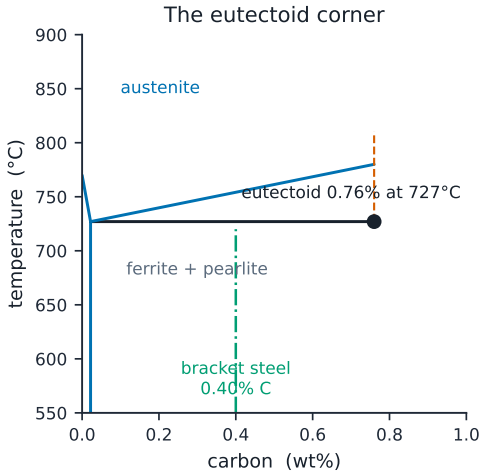
Common mistakes. Mixing engineering and true stress across a table (state the column). Reading yield off a curve with no offset convention stated. Treating toughness (area) as strength (height) – one is joules, the other pascals.

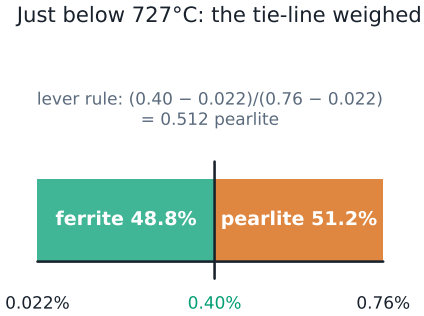
Summary. $\sigma = F/A_0$; E steel/aluminium/nylon = 200/70/2.0 GPa; the rod: 191.0 MPa, 0.477 mm vs 1.364 mm. True $= \ln(1+e)$, 1.2σ at $e = 0.20$. Toughness 176 MJ/m^3 – the joules budget.

Metals and alloys

Why it matters. The steel bracket's final properties are written by chemistry and thermal history – the same steel ships at four different strength/ductility pairs.

The idea, with a picture. The iron-carbon **phase diagram** reads composition and temperature into phases; just below 727°C, the eutectoid reaction turns one austenite grain into lamellar pearlite – and how fast you cool past that line decides everything.





*Read the phases off the diagram; weigh them with the lever rule

The definitions. The **eutectoid** point (0.76 wt% C, 727°C) is where austenite decomposes to **pearlite** (ferrite + cementite lamellae). Slow cooling gives coarse pearlite (soft), quenching bypasses pearlite to **martensite** (hard, brittle, must be tempered). The **lever rule** weighs phase fractions on a tie-line. **Strain hardening** raises strength by cold work: $\sigma = K\varepsilon^n$. Alloying adds **precipitates** that pin dislocations (the 6061 aluminium's Mg_2Si).

The derivation. The bracket steel at 0.40 wt% C, just below 727°C: pearlite fraction $= (C_0 - C_\alpha)/(C_e - C_\alpha) = (0.40 - 0.022)/(0.76 - 0.022) =$ **0.512**, ferrite **0.488** – they sum to one ✓ (the picture's right panel is exactly this tie-line). Cold work: with $K = 530$ MPa and $n = 0.20$, a 25% cold reduction carries true stress $530 \times 0.25^{0.20} =$ **401.7** MPa – the annealed 250 MPa becomes a worked 420 MPa *engineering* yield, but elongation falls 28% → 12%.

> **Worked example 1 – pick the state, not just the grade.** The mount bracket needs formability in production and strength in service. Buy the steel annealed (250 MPa, 28% elongation), form it, then work-harden or heat-treat to the service state – the certificate names *both* states, because the part that ships is the processed one.

> **Worked example 2 – escalate to the quench window.** The eutectoid temperature sets the heat-treat ceiling: austenitise above 727°C (our 0.40% steel: into the full-austenite field), quench fast enough to miss the nose of the transformation curve, temper to trade hardness back for toughness. The 200°C temper ships 950 MPa at 8% elongation; the 600°C temper ships 620 MPa at 16% (the full menu is unit 4.5's picture) – the certification picks one line of that menu.

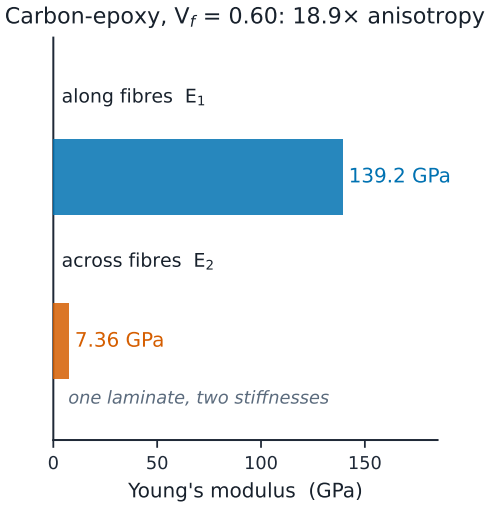
Common mistakes. Reading phases above the diagram's region (ferrite stops at 0.022% below 727°C). Adding strength mechanisms arbitrarily – they interact, so the final certificate is measured, not summed. Confusing cementite (a phase) with pearlite (a mixture).

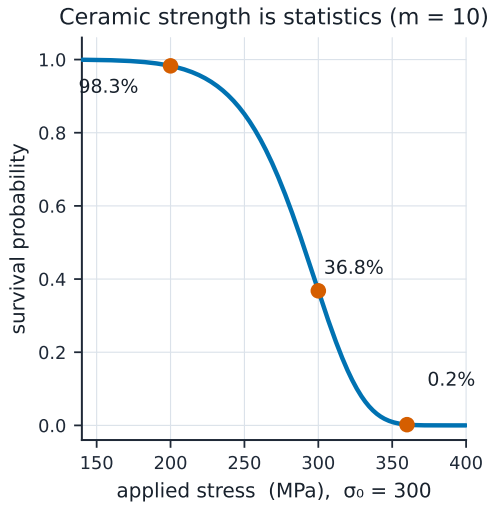
Summary. Eutectoid 0.76% C at 727°C; 0.40% steel just below: 51.2% pearlite + 48.8% ferrite. $\sigma = K\varepsilon^n$: 401.7 MPa true at 25% work. The state ships, not the grade.

Polymers, ceramics and composites

Why it matters. Half the mount's competitors are not metals: nylon promises silence and lightness, ceramic promises stiffness, and the laminate promises both – with catches that are the unit.

The idea, with a picture. The laminate's two moduli – along and across the fibres – and the ceramic's strength statistics: two pictures that carry the whole non-metal story.





*Fibres are one-way promises; ceramics are statistics

The definitions. A **composite** combines phases across a visible scale; the **rule of mixtures** averages properties by volume fraction. The equal-strain model estimates longitudinal stiffness by a weighted arithmetic mean; the equal-stress model gives a weighted harmonic mean. These idealised models provide bounds, not an exact general transverse modulus. **Viscoelasticity** is the polymer's time-dependent stiffness, governed by the **glass transition** (nylon 6: $T_g = 55$°C). A **ceramic** carries few slip systems: strong in compression, brittle in tension, strength scattered – the **Weibull modulus** m prices the scatter: survival $P_s = \exp[-(\sigma/\sigma_0)^m]$.

The derivation. The laminate, $V_f = 0.60$, $E_f = 230$ GPa, $E_m = 3.0$ GPa: along the fibres $E_1 = 0.60 \times 230 + 0.40 \times 3.0 =$ **139.2** GPa ✓; across them $E_2 =$

$E_f E_m/(V_f E_m + V_m E_f) = 690/93.8 = \mathbf{7.36}$ GPa ✓ – an anisotropy of **18.9**, one material with two stiffnesses. The ceramic, $m = 10$, $\sigma_0 = 300$ MPa: $P_s(200) = e^{-(2/3)^{10}} = \mathbf{0.983}$, $P_s(300) = e^{-1} = \mathbf{0.368}$, $P_s(360) = e^{-1.2^{10}} = \mathbf{0.002}$ – twenty per cent over the characteristic strength erases 98% of the survivors: brittle design is statistics.

> **Worked example 1 – the nylon mount in the festival sun.** The booth reaches 45°C, close to nylon's $T_g = 55$°C: the modulus sags viscoelastically (a factor 2–3 is typical across the transition) and creep under sustained load begins. The certification therefore thermal-cycles the nylon candidate to 55°C and re-reads the spring rate – a mount that is stiff in the morning and soft by afternoon fails the till's 2 mm clearance, whatever the room-temperature datasheet says.

> **Worked example 2 – escalate to the laminate bracket.** Loaded along the fibres, 139.2 GPa at $\rho = 1.60$ Mg/m^3 beats everything in the case; loaded across, 7.36 GPa is nylon-class. The design answer is ply orientation: a $[0/\pm 45/90]$ lay-up trades some axial stiffness for usable transverse and shear response – the certificate quotes a *laminate* modulus per direction, never a single number.

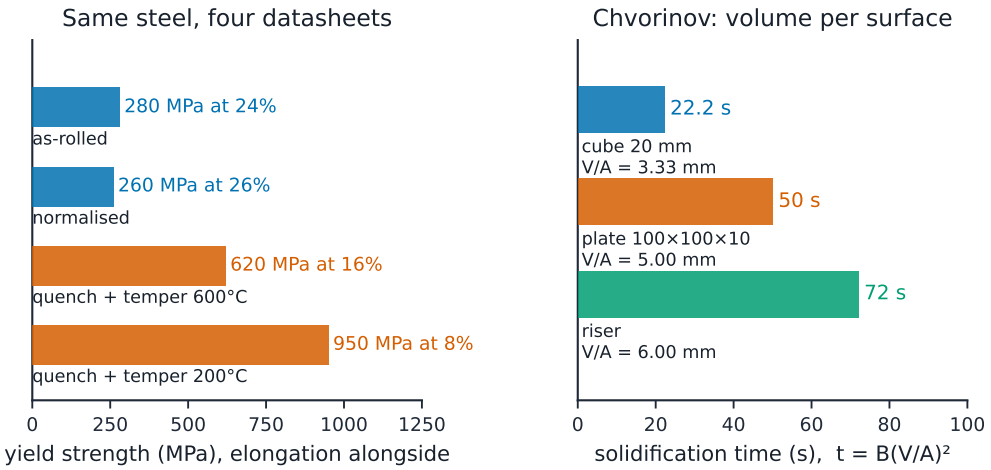
Common mistakes. Using E_1 where the load crosses the fibres (the 19-fold error). Treating the rule of mixtures as exact for strength (fibres fail statistically, not in unison). Designing ceramics to a mean strength instead of a survival probability.

Summary. $E_1 = 139.2$ GPa, $E_2 = 7.36$ GPa, anisotropy 18.9; T_g(nylon) = 55°C – check the sun. Ceramic survival $e^{-(\sigma/\sigma_0)^m}$: 0.983 / 0.368 / 0.002 at 200 / 300 / 360 MPa.

Processing and property links

Why it matters. The menu of bracket properties is written by thermal and mechanical processing; the buyer who ignores it buys a rumour.

The idea, with a picture. Four heat-treatment routes for the same bracket steel, and the casting rule that times solidification: the process map is the property map.



*The furnace writes the datasheet - and the riser sizes the casting

The definitions. Heat treatment – **quenching** from the austenite field to martensite, then **tempering** to give toughness back; the temper temperature dials the trade-off. **Casting** freezes liquid into shape; **solidification** time follows Chvorinov's rule $t = B(V/A)^2$ – volume over cooling surface area, all squared. Work hardening (unit 4.3) is the mechanical member of the same family.

The derivation. The four routes, same 0.40% steel: as-rolled **280** MPa at 24% elongation; normalised **260** at 26%; quenched + tempered 600°C **620** at 16%; quenched + tempered 200°C **950** at 8% ✓ (the picture's left panel – strength triples as ductility halves three times). The casting: with $B = 2.0\times10^6$ s/m^2, a 20 mm cube ($V/A = 3.33$ mm) freezes in $2.0\times10^6\times(3.33\times10^{-3})^2 = \mathbf{22.2}$ s; a $100\times100\times10$ mm plate ($V/A = 5.00$ mm, faces only) takes **50.0** s – 2.25× longer *despite being thinner*, because V/A, not thickness, times the clock ✓ (the ratio is exactly $(5.00/3.33)^2 = 2.25$).

> **Worked example 1 – size the riser.** A casting solidifies last where V/A is largest – so the riser (the reservoir that feeds shrinkage) must have a *bigger* V/A than any section it feeds. The plate's 50.0 s must be beaten: a riser of $V/A = 6.0$ mm gives 72.0 s ✓ and the casting is sound; a riser at $V/A = 4$ mm freezes first and the plate sucks in air – the rule decides, not intuition.

> **Worked example 2 – escalate to the certified route.** The mount bracket's service load is impact (course 3), so ductility matters: the certification picks quenched + tempered 600°C (620 MPa, 16%) over the 200°C temper (950 MPa, 8%) – trading 330 MPa of strength for doubled elongation, because the drop test's joules are absorbed by plastic area, and unit 4.2's toughness is the currency.

Common mistakes. Buying to grade, not state (the menu *is* the material). Tempering too cold for an impact-loaded part (martensite's strength is not the mount's

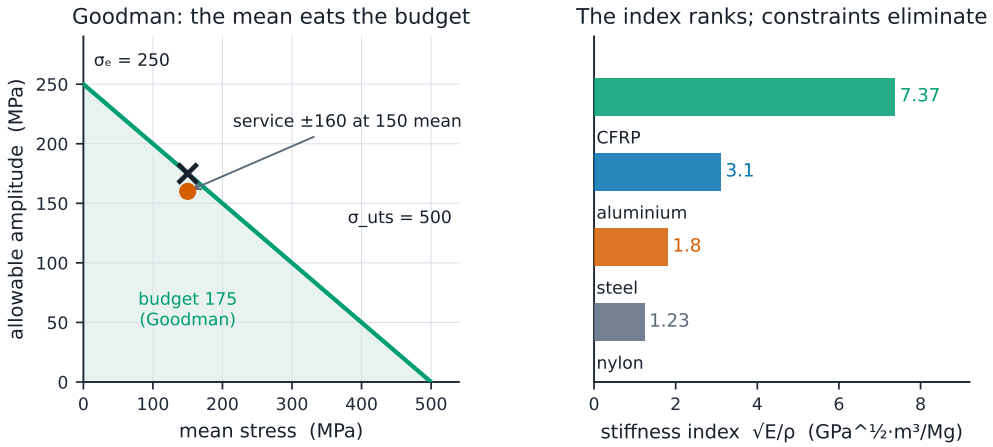
need). Comparing solidification times by thickness instead of V/A.

Summary. Four routes: 280/24, 260/26, 620/16, 950/8 – MPa vs %. Chvorinov $t = B(V/A)^2$: cube 22.2 s, plate 50.0 s, ratio 2.25 = $(V/A)^2$. The riser must be greediest.

Failure and material selection

Why it matters. The mount must survive 10^7 shaker cycles, 25 festival seasons outdoors and one drop – three failure modes, one choice, and a method that shows its arithmetic.

The idea, with a picture. Fatigue's falling curve, corrosion's quiet ledger, and the Ashby index that ranks the four candidates for a stiffness-critical bracket.



*Fatigue prices the cycles; the index ranks the candidates

The definitions. Fatigue fails below yield under repeated load. Basquin's law $\sigma_a = \sigma'_f(2N)^b$ models a stated stress-life range. Many steels show an apparent endurance limit under specified test conditions, while typical aluminium fatigue curves keep falling. An endurance limit is not a guarantee of unlimited life under arbitrary surface, environment and loading conditions. A mean stress eats the budget – the Goodman line $\sigma_a/\sigma_e + \sigma_m/\sigma_{uts} = 1$. **Corrosion** is electrochemical loss, budgeted as thickness per year. A **material index** ranks candidates for a load case: for a stiffness-critical bracket, $\sqrt{E}/\rho$.

The derivation. Fatigue, bracket steel $\sigma'_f = 900$ MPa, $b = -0.10$: at 10^6 reversals $\sigma_a = 900 \times 10^{-0.6} =$ **226.1** MPa; at 10^8, $900 \times 10^{-0.8} =$ **142.6** MPa ✓. With a 150 MPa mean ($\sigma_{uts} = 500$, $\sigma_e = 250$): Goodman allows $\sigma_a = 250(1 - 150/500) =$ **175.0** MPa ✓ – the mean load cost 51 MPa of amplitude. Corrosion: 0.12 mm/yr × 25 yr = **3.0** mm – the wall is bought up front. The index, $\sqrt{E}/\rho$ in GPa$^{1/2}$ · m^3/Mg: steel $\sqrt{200}/7.86 =$ **1.80**, aluminium $\sqrt{70}/2.70 =$ **3.10**, nylon $\sqrt{2.0}/1.15 =$ **1.23**, CFRP $\sqrt{139.2}/1.60 =$ **7.37** ✓ – the laminate wins the stiffness-per-mass game by 2.4× over aluminium.

> **Worked example 1 – the shaker certificate.** Course 3's table cycles at 4.77 Hz between accelerations that load the bracket to ±160 MPa about a 150 MPa mean. Goodman budget is 175 MPa: 160 fits, margin 8.6% – and Basquin extrapolates the life beyond 10^7 reversals at that amplitude ✓. The certificate reads: survives the shaker, 25 seasons – with the arithmetic attached.

> **Worked example 2 – escalate to the full choice.** Constraints: T_g above 60°C (kills plain nylon – 55), corrosion budget 3 mm in steel vs none in CFRP, cost bracket, and the impact toughness of unit 4.2. The index says CFRP (7.37) for stiffness-per-mass; the drop test's joules and the budget say quenched-and-tempered steel. The mount ships as steel; the *carry-handle* – stiffness-critical, light, unloaded in impact – goes CFRP: one method, two right answers, both shown with their index lines.

Common mistakes. Designing fatigue to yield (fatigue lives below it); ignoring the mean stress (Goodman is not optional); comparing index values across *different* load cases (each case has its own exponent); forgetting that the index ranks – constraints eliminate.

Summary. Basquin: 226.1 → 142.6 MPa over $10^6 \to 10^8$; Goodman cuts 250 to 175 at 150 mean. Corrosion: 3.0 mm per 25 yr. Index $\sqrt{E}/\rho$: CFRP 7.37 > aluminium 3.10 > steel 1.80 > nylon 1.23.

The project sheet (reproducible)

The mount's material file, one page:

- Structure: BCC 0.6802, FCC 0.7405; Hall-Petch 183.6 MPa at 50 µm, 262.3 at 10 µm (gain 78.7).
- Elasticity: rod 191.0 MPa → 0.477 mm (steel) vs 1.364 mm (aluminium, ratio 200/70); true strain at 0.20 is ln 1.2 = 0.1823; toughness 176 MJ/m^3.
- Alloys: 0.40% C below 727°C = 51.2% pearlite + 48.8% ferrite; work-hardened 401.7 MPa true at 25%.
- Non-metals: laminate $E_1 = 139.2$, $E_2 = 7.36$ GPa (18.9×); nylon T_g 55°C; ceramic survival 0.983 / 0.368 / 0.002 at 200 / 300 / 360 MPa ($m = 10$).
- Processing: routes 280/24, 260/26, 620/16, 950/8; Chvorinov cube 22.2 s vs plate 50.0 s = $(V/A)^2$.
- Failure: Basquin 226.1 → 142.6 MPa; Goodman 175.0; corrosion 3.0 mm/25 yr; index 7.37 / 3.10 / 1.80 / 1.23 → CFRP handle, steel mount.

Recompute any line from the constants above – the steel ships quenched and tempered at 600°C, the certificate carries Goodman, Basquin and the drop-test budget side by side.

Topic 05

University Mechanical Engineering

What this course is, and how to use it

This is the fifth course in University Mechanical Engineering: **thermodynamics** – energy, entropy and the machinery that turns one into work: how to account for it, how far it can go, and how to know a claim about a machine is impossible before building it.

Scope limit, stated once: a study course, not a degree; the teaching fluids are water, air and R-134a, every cycle is checked by both laws in the course suite, and every table value used here is recomputed there from the quoted source constants.

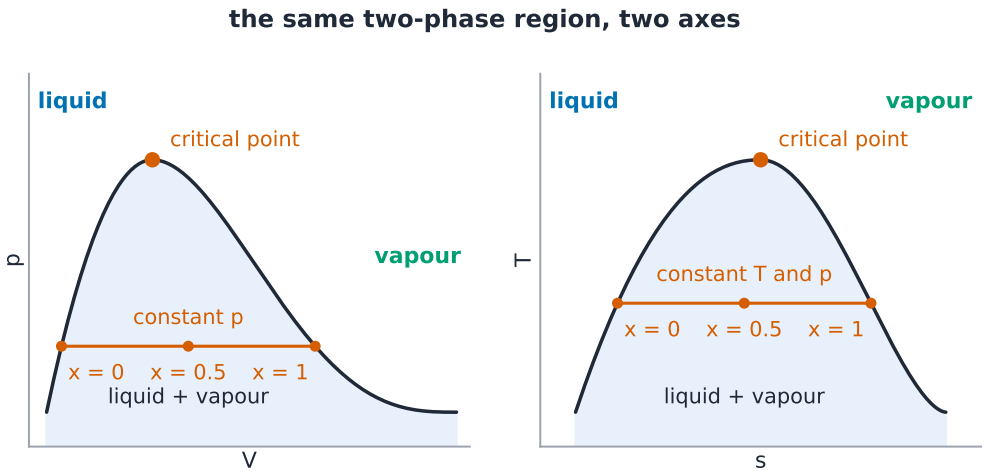
- Units 5.1–5.3 build the accounting: state, closed systems, open systems.
- Unit 5.4 sets the limits: the second law and entropy.
- Units 5.5–5.6 spend the limits: power cycles and refrigeration.

The thermo habit: **every cycle is checked by both laws** – the suite runs the checks; the habit is looking before believing.

Properties and state

Why it matters. Before any law can be applied, the working fluid must be pinned down: what its **state** is, and which numbers move together.

The idea, with a picture. A state is a point on a **property diagram**; two independent intensive properties fix it. On the **saturation dome** the two-phase region sits under the dome; **quality** x = vapour mass fraction walks across it.



Left: the p–V dome, liquid to its left and vapour to its right, with one tie-line marked in quality x; right: the T–s dome of the same substance – the same two-phase region, drawn on different axes

The derivation (why two properties fix the rest). A simple compressible substance obeys a two-variable equation of state, $p = p(v, T)$, so any other equilibrium property is a function of the same pair – pick two independent ones (inside the dome, x is the walker) and everything else is determined.

> **Worked example 1 – reading a table row.** Saturated water at $T = 100\,°\text{C}$: $p_{sat} = 101.35$ kPa, $v_f = 0.001044$ m^3/kg, $v_g = 1.6729$ m^3/kg, $h_f = 419.04$ kJ/kg, $h_{fg} = 2257.0$ kJ/kg. At $x = 0.5$: $v = v_f + x(v_g - v_f) = 0.837$ m^3/kg, $h = h_f + x\,h_{fg} = 1547.5$ kJ/kg – the linear-in-x rule holds for v, u, h, s because dome properties differ only by the latent term.

> **Worked example 2 – ideal-gas territory.** Air at 300 K, 101.325 kPa: $pv = RT$ with $R = 0.287$ kJ/(kg K) gives $v = RT/p = 0.8497$ m^3/kg. At 2× that pressure, same T: $v = 0.4248$ m^3/kg – exactly half: the isotherm is a hyperbola (suite-checked both ways).

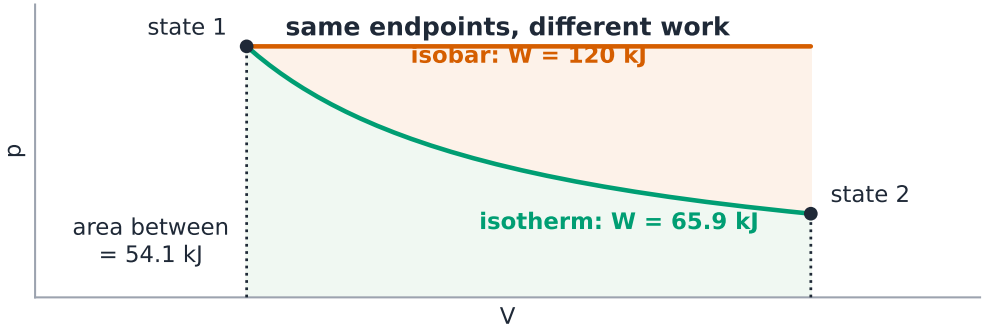
Common mistakes. Using v averaged by volume instead of mass (quality is a mass fraction). Reading a superheated row from a saturation column. Treating $pv = RT$ inside the dome (ideal gas ends where condensation begins).

Summary. Two independent properties fix the state; the dome needs x as the walker; linear-in-x between f and g; $pv = RT$ only outside the dome.

First law for closed systems

Why it matters. The first law is the accountant: energy changes by work and heat crossing the boundary, nothing else, and a closed system is the simplest ledger to keep.

The idea, with a picture. $\Delta E = Q - W$ (work **by** the system positive). On a **p–V diagram** the **boundary work** $W = \int p\,dV$ is the area under the path – so the work depends on the path, not just the endpoints.



Two paths between the same states: the area under each is that path's work – path dependence made visible

The derivation (why work is path-dependent). $W = \int_1^2 p\,dV$ integrates the path $p(V)$ itself, and different connective paths trace different curves between the same endpoints – only U (and H, S) are state functions, the whole reason a cycle's net $\oint \delta W = \oint \delta Q$ closes.

> **Worked example 1 – three paths, one pair of states.** Air from $(200\ \text{kPa}, 0.3\ \text{m}^3)$ to $(200\ \text{kPa}, 0.9\ \text{m}^3)$ versus via an isotherm through the same ends. At constant p: $W = p\Delta V = 200 \times 0.6 = 120$ kJ. Isothermal from $v_1 = v$ to $3v$: $W = mRT\ln(v_2/v_1) = p_1 V_1 \ln 3 = 200 \times 0.3 \times 1.0986 = 65.9$ kJ – same endpoints, 54 kJ apart, the area between the curves is exactly the difference (suite-asserted).

> **Worked example 2 – the adiabatic check.** Same start, compressed adiabatically ($Q = 0$) to $0.15\ \text{m}^3$: $pV^\gamma = \text{const}$ with $\gamma = 1.4$ gives $p_2 = 200 \times (0.3/0.15)^{1.4} = 527.8$ kPa; $W = (p_2 V_2 - p_1 V_1)/(1-\gamma) = (79.2 - 60)/(1 - 1.4) = 48$ kJ of work **on** the gas; $\Delta U = -W = +48$ kJ – the ledger closes.

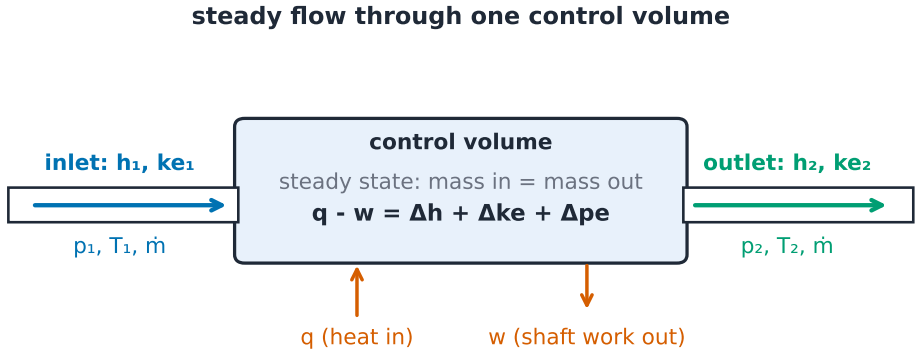
Common mistakes. Sign errors on W (by vs on). Using $p_1 V_1 \approx mRT$ with V in litres and R in kJ. Forgetting that $\Delta U = mc_v\Delta T$ holds for ideal gas even when the process is not constant-volume – it is a property relation.

Summary. $\Delta U = Q - W$; boundary work is area under the path; state functions versus path functions; the ideal-gas toolkit ($pv = RT$, c_v, c_p, pV^γ for reversible adiabatic).

First law for open systems

Why it matters. Real machines are pipes, not pistons: mass flows through a **control volume** and carries energy with it, and steady-state analysis is where turbines, pumps and nozzles live.

The idea, with a picture. The steady-flow energy equation per inlet/outlet port: $q - w = \Delta h + \Delta ke + \Delta pe$. The flow work pv hides inside $h = u + pv$.



One inlet, one outlet, steady state: enthalpy in, enthalpy out, shaft work and heat at the ports

The derivation (where pv goes). Pushing 1 kg of fluid into the control volume costs pv (pressure times volume) of flow work; the same term reappears pushing it out. Combining that transport term with internal energy gives enthalpy – which is why the open-system ledger runs on h, not u: the bookkeeping absorbed the flow work.

> **Worked example 1 – a turbine stage.** Steam enters at 3 MPa, 400 °C ($h_1 = 3230.9$ kJ/kg, $s_1 = 6.9212$ kJ/(kg K)) and expands adiabatically to 50 kPa. Ideal (**isentropic**): $s_2 = s_1$ gives $x_2 = (6.9212 - 1.0910)/6.5029 = 0.89655$, so $h_{2s} = h_f + x\,h_{fg} = 340.49 + 0.89655 \times 2305.4 = 2407.4$ kJ/kg; $w_s = h_1 - h_{2s} = 823.5$ kJ/kg. With $\eta_t = 0.85$: $w = 0.85 \times 823.5 = 700.0$ kJ/kg, $h_2 = 3230.9 - 700.0 = 2530.9$ – the suite recomputes both legs.

> **Worked example 2 – a pump and a nozzle.** Pump lifts liquid water 0.5 MPa → 3 MPa: $w_p \approx v\,\Delta p = 0.001005 \times 2500 = 2.5$ kJ/kg. Nozzle accelerates air 10 → 300 m/s adiabatically, no work: $\Delta h = -\Delta ke = -(300^2 - 10^2)/2 = -44.95$ kJ/kg, $h_2 = h_1 - 44.95$. Three devices, one equation, the ports carry the ledger.

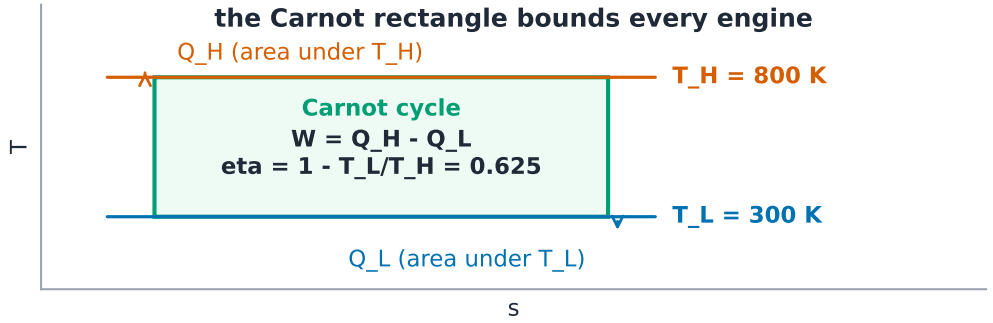
Common mistakes. Dropping ke for a nozzle (it is the point of the device) or keeping it for a pump (negligible there). Using $c_p\Delta T$ for the exit of a two-phase expansion (inside the dome h is not $c_p\Delta T$ from a superheated start). Forgetting $\dot{W} = \dot{m}\,w$ for the power answer.

Summary. $q - w = \Delta h + \Delta ke + \Delta pe$; enthalpy absorbs the flow work; ke matters where velocity is the product; isentropic then efficiency is the two-step turbine method.

Second law and entropy

Why it matters. The first law only balances; the second law forbids. Entropy is the accounting quantity of the forbidden, and the Carnot bound stops bad machine claims on paper.

The idea, with a picture. $ds = \delta q_{rev}/T$ – **entropy** ; $\sigma \geq 0$ for every process, $= 0$ only for reversible ones. On a **T–s diagram** T–s the reversible heat is area under the curve, and the $\eta_C = 1 - T_L/T_H$ appears as the rectangle ratio.



Heat as area on the T–s plane; the Carnot rectangle between T_H and T_L bounds every engine

The derivation (Kelvin–Planck to Carnot). No cycle converts heat from one reservoir fully into work; so for a reservoir pair the best cycle must reject the least, and a reversible one must be the best (otherwise running it backwards would break the law). The reversible pair of isotherms and adiabats – Carnot – then has $Q_H/T_H = Q_L/T_L$ exactly, giving $\eta_C = 1 - T_L/T_H$. Between 800 K and 300 K: $\eta_C = 0.625$; a claimed 0.70 engine between them is impossible, no prototype needed.

> **Worked example 1 – entropy balance on an irreversible adiabatic valve.** Steam at 1000 kPa, $h = 3059$ kJ/kg, $s = 7.155$ kJ/(kg K) throttled to 200 kPa ($h_2 = h_1$): at 200 kPa, $h = 3059$ is superheated, $s_2 = 7.289$ kJ/(kg K); $\sigma = s_2 - s_1 = 0.134$ kJ/(kg K) > 0 – irreversible with no heat transfer at all.

> **Worked example 2 – the Carnot ledger.** A Carnot engine takes 1000 kJ from 800 K, rejects to 300 K: $W = 625$ kJ, $Q_L = 375$ kJ, $\Delta S_{hot} = -1.250$, $\Delta S_{cold} = +1.250$: universe unchanged – reversible. Burn the same fuel irreversibly at 2000 K before transferring: $\sigma = Q(1/800 - 1/2000) = 0.75$ kJ/K – the lost work $T_0\sigma = 300 \times 0.75 = 225$ kJ, exactly the exergy destroyed.

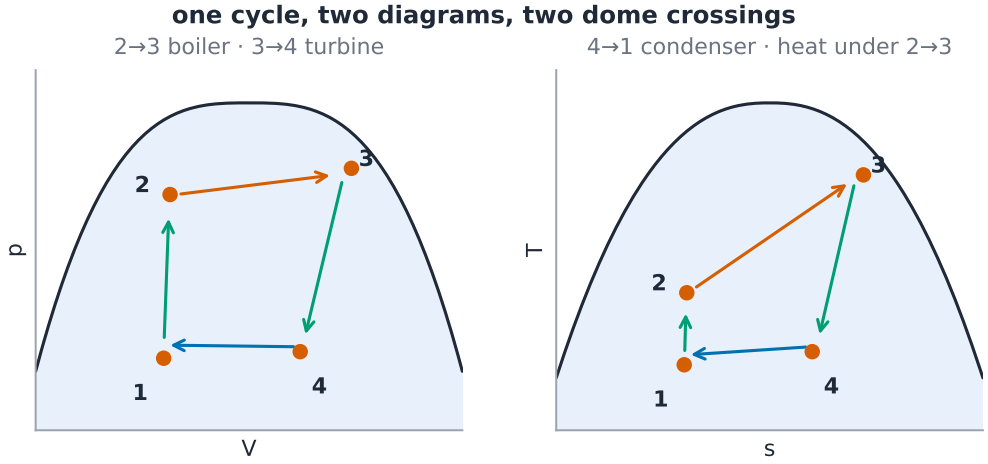
Common mistakes. $ds = \delta q/T$ applied to an irreversible path (it needs q_{rev}, or use the entropy balance with σ). Reading efficiency claims without the reservoir temperatures. Taking s constant across a throttling (throttle keeps h, not s).

Summary. $ds = \delta q_{rev}/T$; $\sigma \geq 0$; Carnot bounds by reservoir ratio; lost work $= T_0\sigma$; T–s area = reversible heat.

Power cycles

Why it matters. The power plant is where the two laws pay rent: the Rankine cycle converts fuel to shaft work through phase change, and its efficiency is a battlefield of boiling pressure, superheat and reheat.

The idea, with a picture. The basic **Rankine cycle** : pump → boiler → turbine → condenser, drawn on both p–V and T–s. Heat in at the high-pressure isobar, work out at the isentropic expansion, heat out at the condenser, work in at the pump.



The same Rankine cycle on both diagrams: the dome crossings at the turbine exhaust and the pump inlet are what make it a two-phase machine

The derivation (why the boiler pressure raises efficiency). The cycle heat-in average temperature rises with boiler pressure, and the second-law ceiling $\eta \leq 1 - T_{out}/T_{in,avg}$ rises with it. Meanwhile the pump work grows only as $v\Delta p$ (liquid v is tiny), so the gain outruns the cost until metallurgy stops it. The suite demonstrates: the same cycle at 1 MPa versus 4 MPa moves η from 25.3% to 32.3% with the same T_{max}.

Worked example 1 – the ideal Rankine baseline. Boiler at 4 MPa, saturated vapour ($h_3 = 2801.4$ kJ/kg, $s_3 = 6.0700$); condenser at 10 kPa. Turbine: $s_4 = s_3$ gives $x_4 = (6.0700 - 0.6492)/7.5009 = 0.7227$, $h_4 = 191.81 + 0.7227 \times 2392.8 = 1921.1$. Pump: $h_2 = h_1 + v_f(p_2 - p_1) = 191.81 + 0.001010 \times 3990 = 195.8$. Work out $2801.4 - 1921.1 = 880.3$, work in 4.0; heat in $2801.4 - 195.8 = 2605.6$; $\eta = 876.3/2605.6 = 33.6\%$ – every step suite-verified.

Worked example 2 – two upgrades, priced. Superheat to 4 MPa, 480 °C ($h_3 = 3398.3$, $s_3 = 6.7677$): $x_4 = 0.8157$, $h_4 = 2143.6$; η rises to 39.1%, and the exhaust quality climbs from 0.72 toward 0.82 – better, though still wet: real plant runs superheat **and** reheat to stay dry. Reheat from an intermediate 800 kPa back to 480 °C pushes the exhaust above 0.9, clearing the 0.88 blade-erosion line.

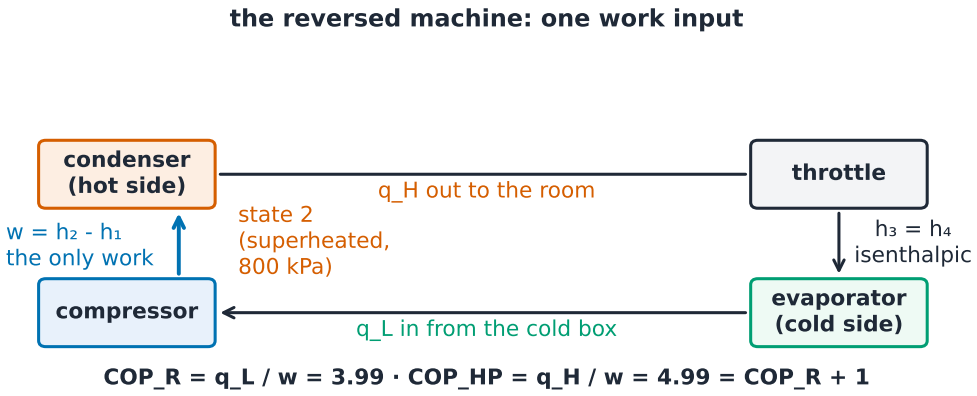
Common mistakes. Subtracting h_1 instead of h_2 in the heat-in term (the pump outlet is where heat starts). Setting $s_2 = s_1$ across the pump (it nearly is, but use $h_2 = h_1 + v\Delta p$). Ignoring exhaust quality – a fine cycle can still ruin blades.

Summary. Four components, two dome crossings; isentropic turbine, liquid pump; superheat and reheat buy efficiency and dry exhaust; boiler pressure raises T_{avg} in.

Refrigeration and heat pumps

Why it matters. Running the cycle backwards moves heat uphill – one skeleton serves the refrigerator and the heat pump alike.

The idea, with a picture. The **vapor-compression cycle** : compressor → condenser → throttle → evaporator. Performance is $\text{COP}_R = q_L/w_{in}$ or $\text{COP}_{HP} = q_H/w_{in}$. The reversed-**Carnot** bounds are $\text{COP}_R \le T_{cold}/(T_{hot} - T_{cold})$ for cooling and $\text{COP}_{HP} \le T_{hot}/(T_{hot} - T_{cold})$ for heating. Use absolute temperatures in kelvin.



The reversed Rankine: condenser above the dome, throttle replacing the pump, evaporator below – the compressor is the only work input

The derivation (why the throttle, not a turbine). The expansion could recover work – but the flow is wet and two-phase turbines are not worth it; a throttle is cheap, isenthalpic, destroys only a few kJ/kg, and the compressor simply re-pays them.

Worked example 1 – an R-134a refrigerator. Evaporator −10 °C ($h_1 = 394.6$ kJ/kg, $s_1 = 1.729$ kJ/(kg K)); compressor to 800 kPa isentropically: $h_2 = 432.9$ (superheated). Condenser exit saturated liquid $h_3 = 241.8$; throttle keeps $h_4 = 241.8$. Then $q_L = h_1 - h_4 = 152.8$, $w_{in} = 38.3$, $\text{COP}_R = 152.8/38.3 = 3.99$. Reversed-Carnot between −10 and 31.3 °C: bound $= 263.2/(304.5 - 263.2) = 6.37$ – real COP sits below, as it must.

Worked example 2 – the same machine heating. As a heat pump, $q_H = h_2 - h_3 = 432.9 - 241.8 = 191.1$ and $\text{COP}_{HP} = 191.1/38.3 = 4.99 = \text{COP}_R + 1$ ✓ – the same numbers read at different ports (suite-asserted).

Common mistakes. COP $= w/q$ (inverted – COP is benefit over cost). Applying the reversed-Carnot bound with temperature differences instead of absolute temperatures. Reading h_2 from the saturated-vapour row (the exit is superheated).

Summary. Reverse the skeleton, swap turbine for throttle; the compressor is the only work; COP is benefit over cost under the reversed-Carnot ceiling, with $\text{COP}_{HP} = \text{COP}_R + 1$.

Topic 06

University Mechanical Engineering

What this course is, and how to use it

This is the sixth course in University Mechanical Engineering: **mechanics of materials** – what the load does to the part: stress, deformation and the failure conditions that decide whether a member is safe, computed on the bracket assembly the track has carried since Course 2.

Scope limit, stated once: a study course, not a degree; members are straight rods, shafts and beams of steel ($E = 200$ GPa, $G = 80$ GPa, $\sigma_y = 250$ MPa), and every number printed here is recomputed in the course suite before it ships.

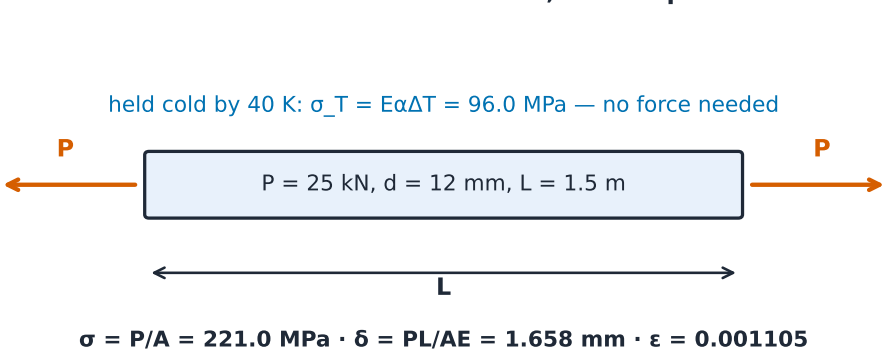
- Units 6.1–6.2 load a member along its axis and about it: axial and torsion.
- Units 6.3–6.4 load it across: the diagrams, then the stresses and the sag they price.
- Unit 6.5 rotates the stress state; unit 6.6 asks when the member stops being one: buckling and failure criteria.

The habit: **compute the stress state, then check it against the limit** – the suite runs both steps on every member.

Axial loading

Why it matters. The tie rod that anchors the bracket is the simplest member in the assembly – and it carries every trap: stress, strain, thermal effects and static indeterminacy in one dimension.

The idea, with a picture. Normal stress $\sigma = P/A$, strain $\varepsilon = \delta/L$, Hooke $\sigma = E\varepsilon$, and **Poisson's ratio** $\nu = 0.30$ squeezing laterally: $\varepsilon_{lat} = -\nu\varepsilon$.



A tie rod stretched by P: the elongation builds linearly along L

The derivation (why $\delta = PL/AE$). Substitute Hooke into the definition: $\delta = \varepsilon L = \sigma L/E = PL/(AE)$ – one line, and the member's stiffness AE/L falls out of it.

> **Worked example 1 – the tie rod.** Steel rod $d = 12$ mm, $L = 1.5$ m, $P = 25$ kN tension: $A = \pi(0.006)^2 = 1.131\times10^{-4}$ m^2, so $\sigma = 25\times10^3/1.131\times10^{-4} = \mathbf{221.0}$ MPa – below $\sigma_y = 250$, safe. The elongation $\delta = PL/(AE) = 37500/(2.262\times10^7) = \mathbf{1.658}$ mm, and the strain $\varepsilon = 0.001105$ squeezes the diameter by $0.30 \times 0.001105 = 0.000332$, i.e. $\mathbf{3.98}$ μm with $\nu = 0.30$.

> **Worked example 2 – held cold.** The same rod fitted between rigid walls and cooled by $\Delta T = 40$ K ($\alpha = 12 \times 10^{-6}$/K) cannot contract, so $\sigma_T = E\alpha\Delta T = 200 \times 10^9 \times 12 \times 10^{-6} \times 40 = \mathbf{96.0}$ MPa tension – a load with no external force, the statically indeterminate signature.

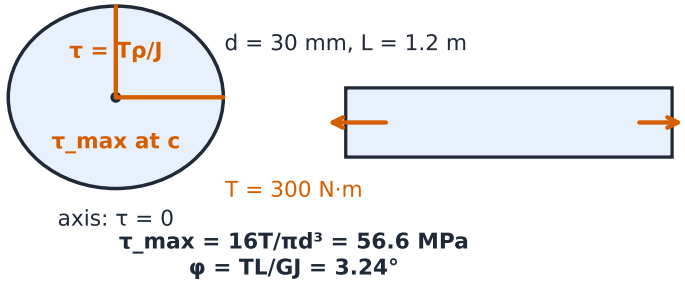
Common mistakes. Using nominal diameter where the thread's root area governs. Adding thermal strain without the constraint that turns it into stress. Forgetting the lateral strain is negative.

Summary. $\sigma = P/A$; $\delta = PL/AE$; $\sigma_T = E\alpha\Delta T$ only when held; Poisson squeezes sideways.

Torsion

Why it matters. The drive shaft between motor and pump is the assembly's **torsion** member, and its size is set by twist as often as by stress.

The idea, with a picture. For circular shafts, $\tau = T\rho/J$ – linear in radius, worst at the surface – and the angle of twist accumulates as $\phi = TL/(GJ)$.



Shear stress grows linearly from the axis; twist accumulates along L

The derivation (why J is geometric). Twist shears the shaft as a rigid-body rotation: $\gamma = \rho\,\phi/L$, so $\tau = G\gamma = G\rho\phi/L$ – the coefficient of ρ must integrate the torque: $T = \int \rho\tau\, dA = (G\phi/L)J$, giving both formulas at once.

Worked example 1 – the drive shaft. Shaft $d = 30$ mm, $T = 300$ N m: $\tau_{max} = 16T/\pi d^3 = 4800/(8.482 \times 10^{-5}) = \mathbf{56.6}$ MPa. Over $L = 1.2$ m: $\phi = TL/(GJ) = 360/6362 = \mathbf{0.0566}$ rad $= \mathbf{3.24}°$.

Worked example 2 – sizing and power. A $T = 400$ N m shaft at $\tau_{allow} = 60$ MPa needs $d^3 = 16 \times 400 \times 10^3/(\pi \times 60) = 33\,960$ mm^3, so $d = \mathbf{32.4}$ mm. The same torque at 180 rpm transmits $P = T\omega = 300 \times 18.85 = \mathbf{5.65}$ kW – stress sizes the shaft, power sells it.

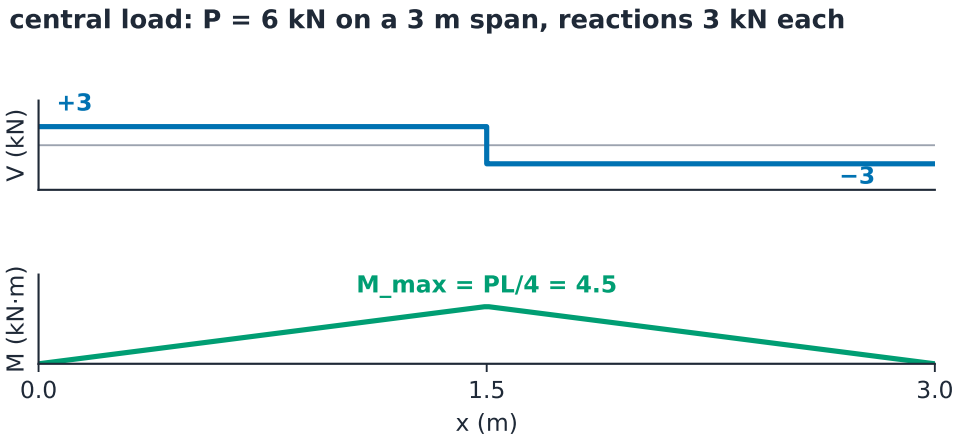
Common mistakes. Using $J = \pi d^4/64$ (that is I) for torsion. Mixing N · mm with MPa · m^4 in $\phi = TL/GJ$. Applying solid-shaft formulas to tubes.

Summary. $\tau_{max} = Tc/J$; $\phi = TL/GJ$; J is $\pi d^4/32$ for solids; power $P = T\omega$.

Shear and bending

Why it matters. Before any stress formula, the beam's internal shear V and moment M must be known along the span – the diagrams are the load's biography, and every formula downstream reads them.

The idea, with a picture. Cut the beam and balance each side: V and M come from equilibrium section by section. Under a downward central $P = 6$ kN on a 3 m simply supported span: reactions 3 kN, V steps $+3 \to -3$ kN at the load, M ramps linearly to its peak $M_{max} = PL/4 = \mathbf{4.5}$ kN m mid-span.



The shear steps, the moment peaks where the shear crosses zero

The derivation (diagrams by the differential relations). Load, shear and moment chain together: $dV/dx = -w(x)$ and $dM/dx = V$ – a point load is the limiting case, stepping V and kinking M; a uniform load w tilts V linearly and bends M parabolically. For $w = 2$ kN/m over the same 3 m span: $V = \pm 3$ kN at the ends and $M_{max} = wL^2/8 = \mathbf{2.25}$ kN m – the same shape family, no single load anywhere.

Worked example 1 – sections, with signs. Sagging is positive: the 3 kN left reaction lifts the left cut, so $V = +3$ kN on the left half and $M = 3x$ grows to **4.5** kN · m at the load – the sign convention is part of the answer, not decoration.

Worked example 2 – read the shape back. Hinges carry no moment ($M = 0$), free ends carry no shear, and M peaks exactly where V crosses zero – reading a diagram backwards recovers the loading.

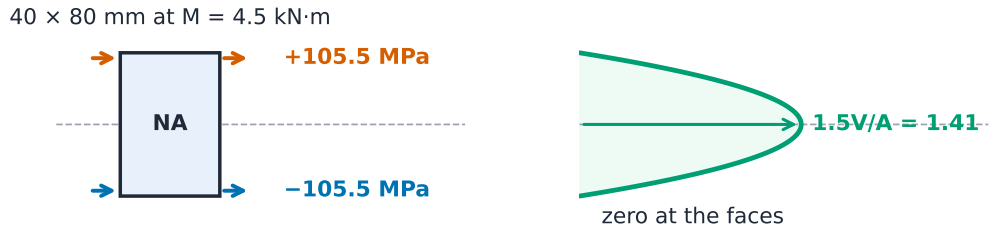
Common mistakes. Dropping the sign convention (sagging vs hogging). Peaking M at a support instead of the $V = 0$ point. Treating a distributed load as a point load at centre for anything but reactions and the peak.

Summary. Cut and balance; $dV/dx = -w$, $dM/dx = V$; central load $M_{max} = PL/4$, uniform load $wL^2/8$; M peaks where $V = 0$.

Beam stresses and deflection

Why it matters. The diagrams price the two beam stresses – bending on the outer fibres, shear on the neutral axis – and the same moment curve, integrated twice, is the sag: the deflection .

The idea, with a picture. The flexure formula $\sigma = Mc/I$ (worst at c, zero at the neutral axis); transverse shear $\tau = VQ/(It)$. In the rectangular section used below, its magnitude peaks at the neutral axis; the distribution depends on section shape.

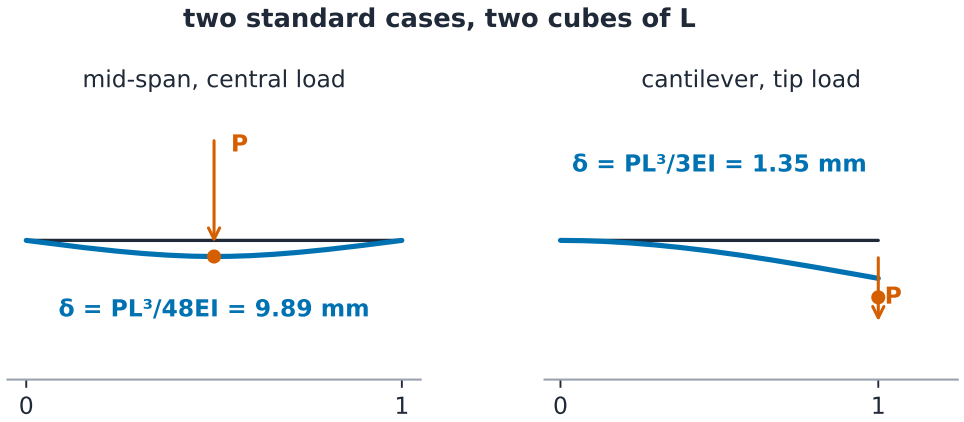


Bending stress linear in depth, transverse shear parabolic – both zero where the other peaks

The derivation (why the rectangle's shear peaks at 1.5V/A). At the neutral axis $Q = bh^2/8$ and $t = b$, so $\tau_{max} = V(bh^2/8)/(b \cdot bh^3/12) = 1.5\,V/A$ – half again the average, the reason short wooden beams fail in shear. For the 40×80 mm arm at $V = 3$ kN: $\tau_{max} = 1.5 \times 3000/0.0032 = \mathbf{1.41}$ MPa.

Worked example 1 – bending stress two ways. With $I = bh^3/12 = \mathbf{1.707 \times 10^{-6}}$ m^4: $\sigma = Mc/I = 180/(1.7067 \times 10^{-6}) = \mathbf{105.5}$ MPa, and the catalogue route $S = I/c = \mathbf{42.7 \times 10^{-6}}$m^3 gives $\sigma = M/S = 4500/(42.7 \times 10^{-6}) = 105.5$ MPa ✓ – same number, one lookup.

Worked example 2 – the sag. The elastic curve $EIv'' = M(x)$ integrates to the standard cases: central load $\delta = PL^3/(48EI) = 162000/(1.638 \times 10^7) = \mathbf{9.89}$ mm mid-span; the same section as a 1.2 m cantilever with 800 N at the tip sags $PL^3/(3EI) = 1382/(1.024 \times 10^6) = \mathbf{1.35}$ mm – and superposition prices combinations: the mid-span sag of (point + uniform) is the sum of the two sags.



Two standard cases: mid-span and cantilever tip

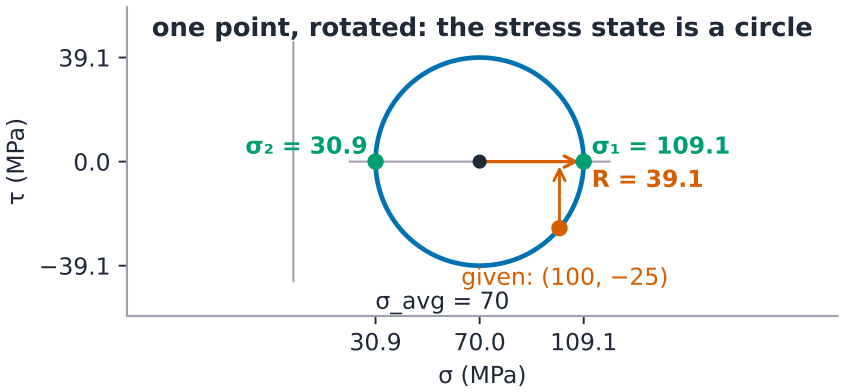
Common mistakes. Full depth for c on a symmetric section. Judging stiffness by stress – a calm beam can still sag past the alignment tolerance (a serviceability limit, checked separately). Forgetting depth enters as h^3: double it, cut the sag eightfold.

Summary. $\sigma = Mc/I = M/S$; $\tau_{max} = 1.5V/A$ for rectangles; $\delta = PL^3/48EI$ and $PL^3/3EI$, superposed when the loads combine.

Combined stress and Mohr's circle

Why it matters. A weld, a shaft point or a gauge location carries stresses at an angle, and the extreme values – not the x-y pair – meet the failure criteria.

The idea, with a picture. Rotation of the stress element is Mohr's circle : centre $\sigma_{avg} = (\sigma_x + \sigma_y)/2$, radius $R = \sqrt{((\sigma_x - \sigma_y)/2)^2 + \tau_{xy}^2}$.



One point's stress state as a circle: intercepts are the principal stresses

The derivation (why the circle). Equilibrium of a wedge cut at angle θ gives the transformation equations; eliminating θ leaves $(\sigma - \sigma_{avg})^2 + \tau^2 = R^2$ – a circle in the σ-τ plane that turns through 2θ as the element turns θ.

Worked example 1 – one point, rotated. A bracket point carries $\sigma_x = 100$, $\sigma_y = 40$, $\tau_{xy} = 25$ MPa: $\sigma_{avg} = 70$, $R = \sqrt{30^2 + 25^2} = \sqrt{1525} = \mathbf{39.1}$ MPa, so the principal stresses $\sigma_1 = 70 + 39.1 = \mathbf{109.1}$ and $\sigma_2 = 70 - 39.1 = \mathbf{30.9}$ MPa sit at $\theta_p = \tan^{-1}(50/60)/2 = 19.9°$ – and the absolute-maximum shear, with both principals positive, is $\sigma_1/2 \approx \mathbf{54.5}$ MPa.

Worked example 2 – opposite signs. Swap the second normal stress to -40 MPa: the centre drops to 40 and $R = \sqrt{80^2 + 30^2} = \mathbf{85.4}$ MPa, so $\sigma_1 = \mathbf{125.4}$, $\sigma_2 = \mathbf{-45.4}$ MPa – now the principals straddle zero and the absolute-maximum shear IS the radius, 85.4 MPa ✓: the sign pattern picks the answer.

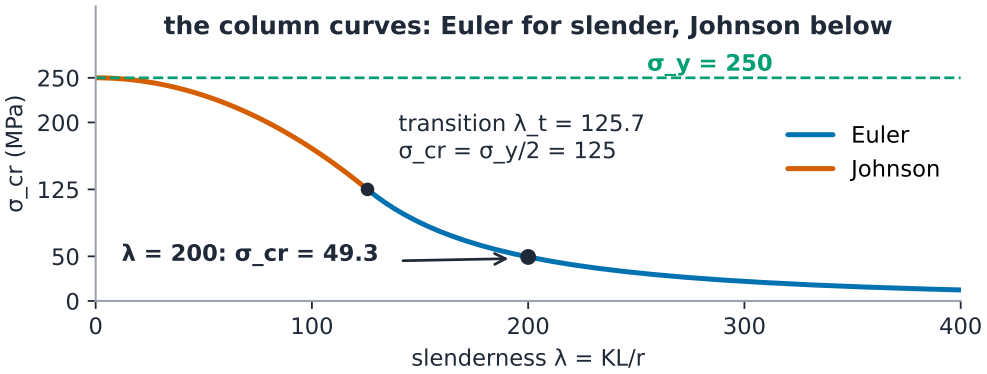
Common mistakes. Halving the wrong angle (the circle turns twice the physical angle). Using in-plane R as the absolute maximum when the principals share a sign. Q as a force – it is a first moment of area.

Summary. Centre σ_{avg}, radius R; principals on the $\tau = 0$ axis; absolute-maximum shear is $\sigma_1/2$ for same-sign principals, R otherwise.

Buckling and failure criteria

Why it matters. A slender push member fails sideways at a fraction of yield, and a computed stress state still needs a criterion to be judged against – buckling and failure criteria close the course.

The idea, with a picture. For an ideal slender elastic column, Euler's load is $P_{cr} = \pi^2 EI/(KL)^2$, with $K = 1$ for pinned ends; below the transition slenderness $\lambda_t = \sqrt{2\pi^2 E/\sigma_y}$, the Johnson parabola $\sigma_{cr} = \sigma_y - (\sigma_y \lambda)^2/(4\pi^2 E)$ takes over, meeting Euler exactly at $\sigma_{cr} = \sigma_y/2$.



Euler for slender columns, Johnson below the transition – yield is the ceiling

The derivation (why Euler divides by L^2). A pinned-pinned column bows into a half-sine; balancing the internal moment EIv'' against $-Pv$ admits that shape only at $P = \pi^2 EI/L^2$ – the longer the member, the softer the spring it becomes, quadratically.

> **Worked example 1 – the strut, with a safety factor.** Steel rod $d = 20$ mm, $L = 1.0$ m pinned-pinned: $I = 7.854 \times 10^{-9}$ m^4, so $P_{cr} = \pi^2 \times 200 \times 10^9 \times 7.854 \times 10^{-9} =$ **15.5** kN, at $\sigma_{cr} =$ **49.3** MPa. Slenderness $\lambda = KL/r = 1000/5 =$ **200** – well past $\lambda_t =$ **125.7**, so Euler applies; at $\lambda = 60$ Johnson gives $250 - 28.5 =$ **221.5** MPa. At a design factor $n = 2.5$: $P_{allow} = P_{cr}/n =$ **6.2** kN.

> **Worked example 2 – judging the stress state.** The rod under eccentric loading : 30 mm, $P = 5$ kN at $e = 60$ mm carries axial 7.07 plus bending $Pec/I = 300 \times 0.015/(3.976 \times 10^{-8}) =$ **113.2** MPa, peaking at **120.3** MPa on the far fibre. Von Mises on the Mohr example's principals $(109.1, 30.9)$: $\sigma' = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2} =$ **97.4** MPa – under the 250 MPa yield, while max-shear reads $\tau_{max} =$ **39.1** against its $\sigma_y/2 =$ **125** MPa limit: both criteria agree, no yield.

Common mistakes. Applying Euler below λ_t (it then exceeds yield – impossible). Fixing K by habit instead of the end conditions. Using the failure criterion on the x-y stresses instead of the principals. quoting an allow load without its factor.

Summary. $P_{cr} = \pi^2 EI/(KL)^2$ for $\lambda > \lambda_t$, Johnson below; design at P_{cr}/n; judge states by von Mises σ' or max shear on the principals.

exercises.lol exercises.com.cn