Algebra & Graphs

IGCSE Mathematics ▪ Topic 2

IGCSE Mathematics

What we will cover

1. Simplify, expand & factorise
2. Solving equations
3. Quadratics & inequalities
4. Sequences
5. Graph shapes
6. Travel graphs

1 Manipulating algebra

Simplify and expand

	x	2
3	$3x$	6

$3(x+2) = 3x+6$

In **algebra** 代数 we use letters to stand for numbers.

> A letter whose value can change is a **variable** 变量.

The words an algebra question uses

Expression and constant	An **expression** 表达式 has no equals sign; a **constant** 常数 is a fixed number in it, as against a variable.
Substitute	To **substitute** 代入 is to replace each letter by its value and evaluate – brackets first, and mind the signs.
Algebraic fraction	An **algebraic fraction** 代数分式 (a **rational expression** 有理式) is one polynomial over another; simplify by factorising both and cancelling common factors.
Fractional equation	A **fractional equation** 分式方程 has the unknown in a denominator: multiply through by the common denominator, then check no solution makes it zero.

> A **perfect square** 完全平方 factorises as $(x+a)^2$ – spotting one saves the whole method of completing the square.

Factorising

common factor	**difference of squares**	**quadratic**
$9x^2 + 15xy$ $= 3x(3x+5y)$	$a^2 - b^2$ $= (a+b)(a-b)$	split the middle term, then group

Factorise 因式分解 is the reverse of expanding: take out the common factor first.

Solving equations

$2x + 3 = 11$ (start)
$2x = 8$ subtract 3 from both sides
$x = 4$ divide both sides by 2

Do the same to both sides until the **unknown** 未知数 is alone.

Simultaneous 联立方程
$2x + y = 7$, $3x - y = 8$
add: $5x = 15$, so $x = 3$, $y = 1$

Changing the subject

To **change the subject 公式变形** of a formula means to rearrange it so a chosen letter is alone on one side.

Worked example. Make r the subject of $A = \pi r^2$ (**Extended**, because of the power).

2 Quadratics and more

Quadratic equations

factorise
$(x+2)(x+3) = 0$

complete the square
$(x+3)^2 = 8$

formula
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Three routes to a quadratic's roots

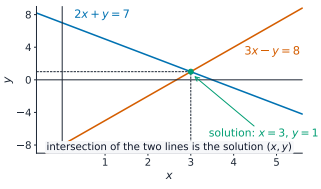
Factorising	Fastest when it works: write as a product of two brackets and set each to zero.
Completing the square	Gives the **maximum** 最大值 or minimum point directly, and the **axis of symmetry** 对称轴 with it.
By the quadratic formula	**By the quadratic formula** 用二次公式: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ – always available, and the discriminant tells you how many roots first.
Reading the graph	The roots are where the curve meets the x-axis; the **intercept** 截距 on the y-axis is the constant term.

An **intersection point** 交点 of two graphs is the solution of their simultaneous equations – and vice versa.

Simultaneous equations

Simultaneous equations 联立方程 are two equations solved together. For two linear equations, add or subtract to remove one letter.

The solution is where the two graphs cross – one point that satisfies both equations at once.



Inequalities

Solve like an equation, but reverse the sign if you $\times$ or $\div$ by a negative. Number line: closed circle includes the value, open excludes it.

$-\frac{1}{3}$ included (closed circle) 3 excluded (open circle)

An **inequality** 不等式 uses $<$, $>$, $\leqslant$ or $\geqslant$.

> Solve it like an equation, but **reverse the sign** if you multiply or divide by a negative number.

Sequences

Find the nth term from how the terms change.

- linear $2, 5, 8, 11 \cdots \to 3n - 1$
- quadratic: $2\times$ the 2nd difference $= n^2$ part

$2, 5, 10, 17 \cdots \to n^2 + 1$.

Sequences, indices in algebra, and the derivative

a sequence 数列	a list of numbers following a rule
the term-to-term rule 递推规则	tells you how to get the next term from the one before
the nth-term rule	gives any term directly from its position – far more useful
indices 指数 in algebra	$a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$
(Extended) the derivative	if $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$
what it gives you	the gradient at any point, so stationary points come from $\frac{dy}{dx} = 0$

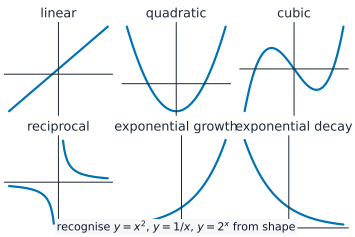
> "Multiply the power by the coefficient, then reduce the power by one." Say it aloud once and the rule stops needing to be looked up.

3 Graphs

Graph shapes

Know each shape: **linear** 一次, **quadratic** 二次 (parabola), cubic, **reciprocal** 反比例 & **exponential** 指数. **Roots** 根 are where the graph crosses the x-axis ($y = 0$).

Graph shapes, in detail



> Know each shape: **linear** 一次, **quadratic** 二次 (parabola), cubic, **reciprocal** 反比例 & **exponential** 指数.

Growth, decay and the line a curve never meets

Exponential growth	**Exponential growth** 指数增长: $y = ab^x$ with $b > 1$ – a fixed *multiplier* each step, not a fixed increase.
Exponential decay	**Exponential decay** 指数衰减: the same form with $0 < b < 1$. An **exponential sequence** 指数数列 is its discrete version, with a common ratio.
Asymptote	An **asymptote** 渐近线 is a line the curve approaches but never reaches – the x-axis for an exponential, both axes for $y = 1/x$.
Direct proportion	**Direct proportion** 正比例 means $y = kx$: a straight line through the origin. Inverse proportion means $y = k/x$, a hyperbola.

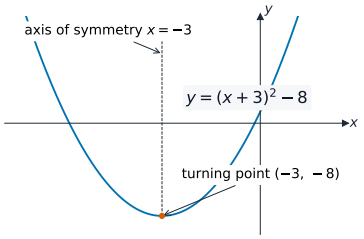
The **base** 底数 b decides everything: above one it grows, below one it decays, and at one it is a horizontal line.

Drawing graphs, and reading a region

a table of values 数值表	choose values of x, work out y, then plot the **coordinates** 坐标
how many points	a straight line needs three (two plus a check); a curve needs five or more
a conversion graph 换算图	a straight line used to change between two units – miles and kilometres
a travel graph	distance–time: the gradient is speed. Speed–time: the gradient is acceleration and the area is distance
(Extended) a region 区域	an inequality in two letters describes a region
drawing it	draw the boundary line – broken for $<$ or $>$, solid for $\leq$ or $\geq$ – then shade

Test the origin in the inequality to decide which side to shade. If $(0, 0)$ satisfies it, shade the side containing the origin.

Completing the square

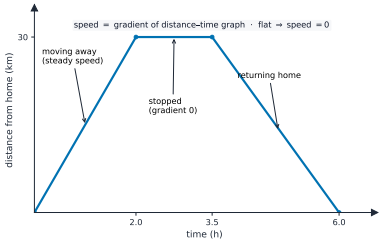


Writing $y = (x+p)^2 + q$ shows the **turning point** 顶点 at $(-p, q)$.

$$x^2 + 6x + 1 = (x+3)^2 - 8$$

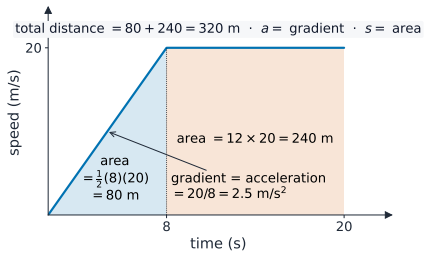
So the lowest point is $(-3, -8)$.

Travel graphs – one



Travel graphs

Travel graphs – two



Travel graphs

Sketching curves

A quick sketch should show the right shape and the key features: where it crosses the axes, any **symmetry** 对称, and any line the curve gets close to.

For a parabola, completing the square gives the **turning point** 转折点 (the lowest or highest point).

Differentiation (Extended)

Differentiation 微分 finds the gradient of a curve. If $y = ax^n$:

$$\frac{dy}{dx} = a\,n\,x^{n-1}$$

differentiate a sum term by term.

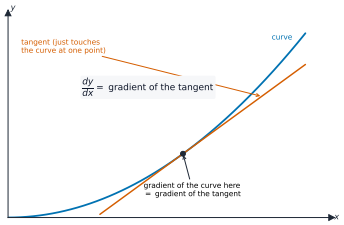
A **turning point** 驻点 is where the gradient is zero – set $dy/dx = 0$ and solve.

Turning point of $y = x^2 - 6x + 5$

$\frac{dy}{dx} = 2x - 6 = 0 \Rightarrow x = 3$

$y = 9 - 18 + 5 = -4$, so $(3, -4)$

Differentiation (Extended)



The gradient of a curve at a point equals the gradient of the tangent there – what differentiation finds.

Functions (Extended)

A **function** 函数 turns each input into one output, $f(x)$.

- **inverse** 反函数 f^{-1} undoes it: swap x and y, solve
- **composite** 复合函数 $gf(x)$ means do f first, then g

$f(x) = 2x$, $g(x) = x + 3$

$gf(x) = g(2x) = 2x + 3$

$fg(x) = f(x + 3) = 2x + 6$

Order matters: $gf \neq fg$.

Direct and inverse proportion (Extended)

- **direct** 正比例: $y = kx$ – one is a fixed multiple of the other
- **inverse** 反比例: $y = k/x$ – one rises as the other falls

Always find k first from the given pair.

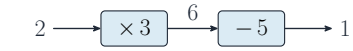
$y \propto x$; $y = 12$ when $x = 3$. Find y at $x = 7$.

$12 = k \times 3 \Rightarrow k = 4$, so $y = 4x$

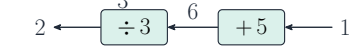
$y = 4 \times 7 = 28$

Functions (Extended)

$f(x) = 3x - 5$: do $\times 3$, then -5



$f^{-1}(x) = \frac{x+5}{3}$: **reverse the order, undo each step**



the inverse takes the output back to the input: $f(2) = 1$, so $f^{-1}(1) = 2$

A function as a machine; the inverse runs it backwards - reverse the order, undo each step

Function words, and rate of change

Domain	The **domain** 定义域 is the set of inputs allowed – restricted wherever a denominator vanishes or a square root would go negative.
Composite function	A **composite function** 复合函数 $fg(x)$ applies g first, then f. The order changes the answer.
Inverse function	An **inverse function** 反函数 undoes f: swap x and y and make y the subject. Its graph is the reflection in $y = x$.
Rate of change	The gradient of a travel graph is a **rate of change** 变化率 – speed on a distance–time graph, and a negative gradient there is a **deceleration** 减速.

An inverse exists only if each output came from one input – which is why a domain is so often restricted first.

Simplifying and expanding

$$3a + 2a + 4b - b$$

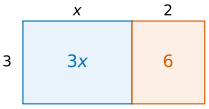
$$\downarrow \text{ collect like terms}$$

$$5a + 3b$$

Collecting like terms: add terms with the same letter

Expand by distributing: a(b+c)=ab+ac (area model)

$3\,(x + 2) = 3x + 6$



$a(b + c) = ab + ac$ · area model of expansion

Expand by distributing: a(b+c)=ab+ac (area model)

$2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$

↓ *group like terms*

$7a^2 - 6ab + 3$

terms that share a colour are like terms

like terms: same letters/powers · add coefficients only

Like terms share the same letters and powers – add coefficients only

Expand by distributing: a(b+c)=ab+ac (area model)

Completing the square (Extended)

A suspension bridge cable hangs in a parabola – the graph of a quadratic.

Exam tips

- When you **expand brackets**, multiply every term and watch the signs, especially with a minus in front: $-(x - 3) = -x + 3$.
- To solve an equation, do the same thing to both sides. When you multiply or divide an **inequality** by a negative number, *flip* the sign.
- **Factorise fully**: take out the highest common factor first, then look for a difference of two squares or a quadratic pattern.
- A **quadratic** usually has two solutions – give both. Check whether the question wants factorising, the formula, or completing the square.
- When substituting into a formula, put each value in brackets first, so signs and powers come out right.

4 Recap

Topic 2 – key takeaways

1 Expand

multiply out; factorise reverses it

2 Quadratics

factorise, complete the square, or formula

3 Inequalities

flip the sign when dividing by a negative

4 Graphs

gradient = rate; area under speed–time = distance

Check yourself

1. Expand $(x+2)(x-4)$.
2. Factorise $9x^2 - 16$.
3. When must you reverse an inequality sign?
4. What does the area under a speed–time graph give?

Answers **1.** $x^2 - 2x - 8$ **2.** $(3x+4)(3x-4)$ **3.** when $\times$ or $\div$ by a negative **4.** the distance travelled