Algebra & Graphs

IGCSE Mathematics ▪ Topic 2

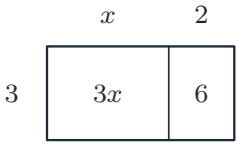
IGCSE Mathematics

What we will cover

1. Simplify, expand & factorise
2. Solving equations
3. Quadratics & inequalities
4. Sequences
5. Graph shapes
6. Travel graphs

1 Manipulating algebra

Simplify and expand



$$3(x+2) = 3x + 6$$

In **algebra** 代数 we use letters to stand for numbers.

> A letter whose value can change is a **variable** 变量.

The words an algebra question uses

Expression and constant	An **expression** 表达式 has no equals sign; a **constant** 常数 is a fixed number in it, as against a variable.
Substitute	To **substitute** 代入 is to replace each letter by its value and evaluate – brackets first, and mind the signs.
Algebraic fraction	An **algebraic fraction** 代数分式 (a **rational expression** 有理式) is one polynomial over another; simplify by factorising both and cancelling common factors.
Fractional equation	A **fractional equation** 分式方程 has the unknown in a denominator: multiply through by the common denominator, then check no solution makes it zero.

A **perfect square** 完全平方 factorises as $(x + a)^2$ – spotting one saves the whole method of completing the square.

Factorising

common factor

$$9x^2 + 15xy = 3x(3x + 5y)$$

difference of squares

$$a^2 - b^2 = (a+b)(a-b)$$

quadratic

split the middle term, then group

Factorise 因式分解 is the reverse of expanding: take out the common factor first.

Solving equations

$$2x + 3 = 11 \quad \text{(start)}$$

$$2x = 8 \quad \text{subtract 3 from both sides}$$

$$x = 4 \quad \text{divide both sides by 2}$$

Do the same to both sides until the **unknown** 未知数 is alone.

Simultaneous 联立方程

$2x + y = 7$, $3x - y = 8$
add: $5x = 15$, so $x = 3$, $y = 1$

Changing the subject

To **change the subject** 公式变形 of a formula means to rearrange it so a chosen letter is alone on one side.

> **Worked example.** Make r the subject of $A = \pi r^2$ (**Extended**, because of the power).

2 Quadratics and more

Quadratic equations

factorise

$$(x+2)(x+3)=0$$

complete the square

$$(x+3)^2=8$$

formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Three routes to a quadratic's roots

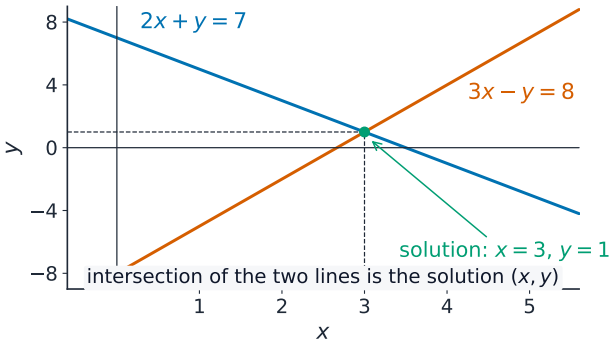
Factorising	Fastest when it works: write as a product of two brackets and set each to zero.
Completing the square	Gives the **maximum** 最大值 or minimum point directly, and the **axis of symmetry** 对称轴 with it.
By the quadratic formula	**By the quadratic formula** 用二次公式: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ – always available, and the discriminant tells you how many roots first.
Reading the graph	The roots are where the curve meets the x-axis; the **intercept** 截距 on the y-axis is the constant term.

An **intersection point** 交点 of two graphs is the solution of their simultaneous equations – and vice versa.

Simultaneous equations

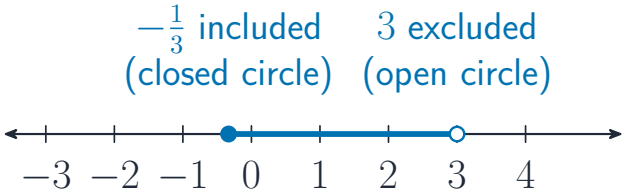
Simultaneous equations 联立方程 are two equations solved together. For two linear equations, add or subtract to remove one letter.

> The solution is where the two graphs cross – one point that satisfies both equations at once.



Inequalities

Solve like an equation, but reverse the sign if you $\times$ or $\div$ by a negative. Number line: closed circle includes the value, open excludes it.



An **inequality** 不等式 uses $<$, $>$, $\leqslant$ or $\geqslant$.

> Solve it like an equation, but **reverse the sign** if you multiply or divide by a negative number.

Sequences

Find the nth term from how the terms change.

- linear $2, 5, 8, 11 \cdots \to 3n - 1$
- quadratic: $2\times$ the 2nd difference $= n^2$ part

$2, 5, 10, 17 \cdots \to n^2 + 1.$

Sequences, indices in algebra, and the derivative

a sequence 数列	a list of numbers following a rule
the term-to-term rule 递推规则	tells you how to get the next term from the one before
the *n*th-term rule	gives any term directly from its position – far more useful
indices 指数 in algebra	$a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$
(Extended) the derivative	if $y = ax^n$ then $\dfrac{dy}{dx} = anx^{n-1}$
what it gives you	the gradient at any point, so stationary points come from $\dfrac{dy}{dx} = 0$

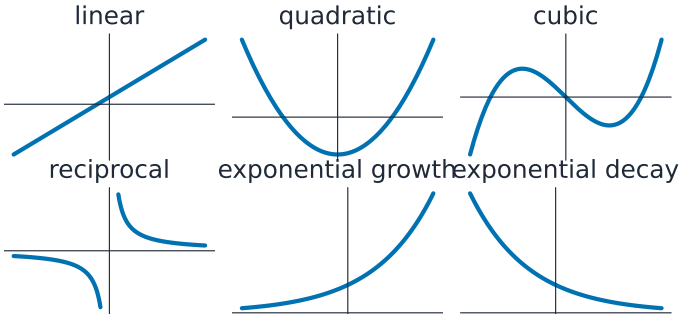
"Multiply the power by the coefficient, then reduce the power by one." Say it aloud once and the rule stops needing to be looked up.

3 Graphs

Graph shapes

Know each shape: **linear** 一次, **quadratic** 二次 (parabola), cubic, **reciprocal** 反比例 & **exponential** 指数. **Roots** 根 are where the graph crosses the x-axis ($y = 0$).

Graph shapes, in detail



recognise $y = x^2$, $y = 1/x$, $y = 2^x$ from shape

Know each shape: **linear** 一次, **quadratic** 二次 (parabola), cubic, **reciprocal** 反比例 & **exponential** 指数.

Growth, decay and the line a curve never meets

Exponential growth

Exponential growth 指数增长: $y = ab^x$ with $b > 1$ – a fixed *multiplier* each step, not a fixed increase.

Exponential decay

Exponential decay 指数衰减: the same form with $0 < b < 1$. An **exponential sequence** 指数数列 is its discrete version, with a common ratio.

Asymptote

An **asymptote** 渐近线 is a line the curve approaches but never reaches – the x-axis for an exponential, both axes for $y = 1/x$.

Direct proportion

Direct proportion 正比例 means $y = kx$: a straight line through the origin. Inverse proportion means $y = k/x$, a hyperbola.

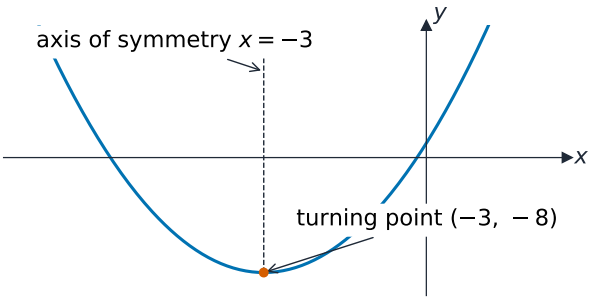
The **base** 底数 b decides everything: above one it grows, below one it decays, and at one it is a horizontal line.

Drawing graphs, and reading a region

a table of values 数值表	choose values of x, work out y, then plot the **coordinates** 坐标
how many points	a straight line needs three (two plus a check); a curve needs five or more
a conversion graph 换算图	a straight line used to change between two units – miles and kilometres
a travel graph	distance–time: the gradient is speed. Speed–time: the gradient is acceleration and the area is distance
(Extended) a region 区域	an inequality in two letters describes a region
drawing it	draw the boundary line – broken for $<$ or $>$, solid for $\leq$ or $\geq$ – then shade

Test the origin in the inequality to decide which side to shade. If $(0, 0)$ satisfies it, shade the side containing the origin.

Completing the square



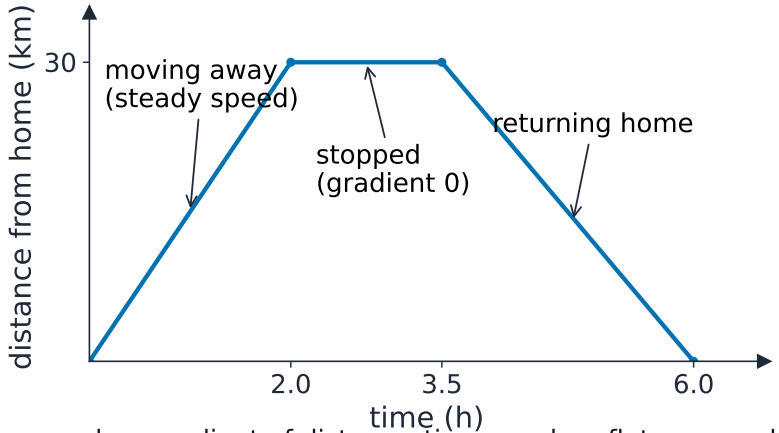
$y = (x + 3)^2 - 8$

Writing $y = (x + p)^2 + q$ shows the **turning point** 顶点 at $(-p, q)$.

$$x^2 + 6x + 1 = (x + 3)^2 - 8$$

So the lowest point is $(-3, -8)$.

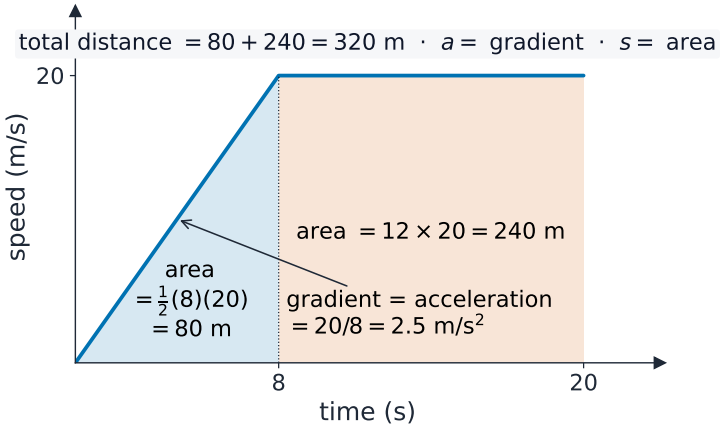
Travel graphs – one



speed = gradient of distance–time graph · flat ⇒ speed = 0

Travel graphs

Travel graphs – two



Travel graphs

Sketching curves

A quick sketch should show the right shape and the key features: where it crosses the axes, any **symmetry 对称**, and any line the curve gets close to.

> For a parabola, completing the square gives the **turning point 转折点** (the lowest or highest point).

Differentiation (Extended)

Differentiation 微分 finds the gradient of a curve. If $y = ax^n$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\,n\,x^{n-1}$$

differentiate a sum term by term.

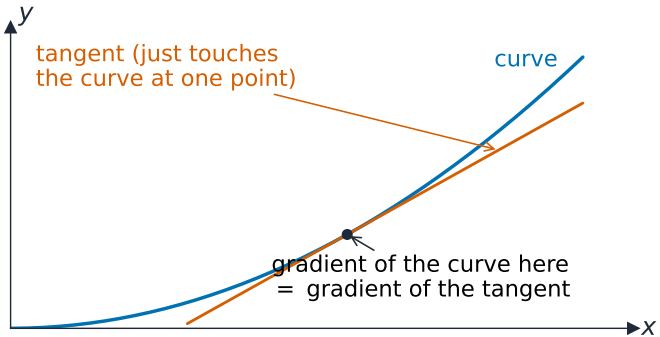
A **turning point** 驻点 is where the gradient is zero – set $\mathrm{d}y/\mathrm{d}x = 0$ and solve.

Turning point of $y = x^2 - 6x + 5$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6 = 0 \Rightarrow x = 3$

$y = 9 - 18 + 5 = -4$, so $(3, -4)$

Differentiation (Extended)



$$\frac{dy}{dx} = \text{ gradient of the tangent}$$

The gradient of a curve at a point equals the gradient of the tangent there – what differentiation finds.

Functions (Extended)

A **function** 函数 turns each input into one output, $f(x)$.

- **inverse** 反函数 f^{-1} undoes it: swap x and y, solve
- **composite** 复合函数 $gf(x)$ means do f first, then g

$f(x) = 2x$, $g(x) = x + 3$

$$gf(x) = g(2x) = 2x + 3$$
$$fg(x) = f(x + 3) = 2x + 6$$

Order matters: $gf \neq fg$.

Direct and inverse proportion (Extended)

- **direct** 正比例: $y = kx$ – one is a fixed multiple of the other
- **inverse** 反比例: $y = k/x$ – one rises as the other falls

Always find k first from the given pair.

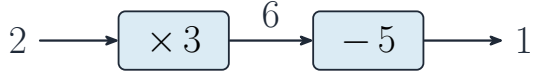
$y \propto x$; $y = 12$ when $x = 3$. Find y at $x = 7$.

$12 = k \times 3 \Rightarrow k = 4$, so $y = 4x$

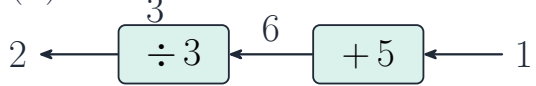
$y = 4 \times 7 = 28$

Functions (Extended)

$f(x) = 3x - 5$: do $\times 3$, then -5



$f^{-1}(x) = \dfrac{x+5}{3}$: **reverse the order, undo each step**



the inverse takes the output back to the input: $f(2) = 1$, so $f^{-1}(1) = 2$

A function as a machine; the inverse runs it backwards - reverse the order, undo each step

Function words, and rate of change

Domain	The **domain** 定义域 is the set of inputs allowed – restricted wherever a denominator vanishes or a square root would go negative.
Composite function	A **composite function** 复合函数 $\mathrm{fg}(x)$ applies g first, then f. The order changes the answer.
Inverse function	An **inverse function** 反函数 undoes f: swap x and y and make y the subject. Its graph is the reflection in $y = x$.
Rate of change	The gradient of a travel graph is a **rate of change** 变化率 – speed on a distance–time graph, and a negative gradient there is a **deceleration** 减速.

An inverse exists only if each output came from one input – which is why a domain is so often restricted first.

Simplifying and expanding

$$3a + 2a + 4b - b$$

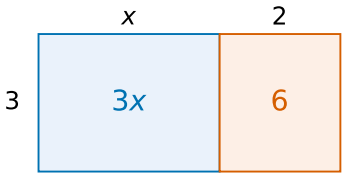
$\downarrow$ collect like terms

$$5a + 3b$$

Collecting like terms: add terms with the same letter

Expand by distributing: a(b+c)=ab+ac (area model)

$$3(x+2) = 3x + 6$$



$a(b+c) = ab + ac$ · area model of expansion

Expand by distributing: a(b+c)=ab+ac (area model)

$$2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$$

↓ *group like terms*

$$7a^2 - 6ab + 3$$

terms that share a colour are like terms

like terms: same letters/powers · add coefficients only

Like terms share the same letters and powers – add coefficients only

Expand by distributing: a(b+c)=ab+ac (area model)

Completing the square (Extended)

A suspension bridge cable hangs in a parabola – the graph of a quadratic.

Exam tips

- When you **expand brackets**, multiply every term and watch the signs, especially with a minus in front: $-(x-3) = -x+3$.
- To solve an equation, do the same thing to both sides. When you multiply or divide an **inequality** by a negative number, *flip* the sign.
- **Factorise fully**: take out the highest common factor first, then look for a difference of two squares or a quadratic pattern.
- A **quadratic** usually has two solutions – give both. Check whether the question wants factorising, the formula, or completing the square.
- When substituting into a formula, put each value in brackets first, so signs and powers come out right.

4 Recap

Topic 2 – key takeaways

1 Expand

multiply out; factorise reverses it

2 Quadratics

factorise, complete the square, or formula

3 Inequalities

flip the sign when dividing by a negative

4 Graphs

gradient = rate; area under speed–time = distance

Check yourself

1. Expand $(x+2)(x-4)$.
2. Factorise $9x^2 - 16$.
3. When must you reverse an inequality sign?
4. What does the area under a speed–time graph give?

Answers **1.** $x^2 - 2x - 8$ **2.** $(3x+4)(3x-4)$ **3.** when $\times$ or $\div$ by a negative **4.** the distance travelled