

1.7 Series, finite differences and Taylor expansions

Name: _____ Class: _____ Date: _____

Total: 13 marks

KEY WORDS Taylor polynomial

Objective

Series, finite differences and Taylor expansions. Supported HL concept preparation; original classroom practice, not an official IB paper or assessed learner work.

1 Key knowledge and worked examples

■ Can a polynomial replace an exponential nearby?

- Near a known point, a complicated function can be replaced by a polynomial approximation. The approximation needs a centre and a valid scale.
 - This lesson studies **Taylor polynomial** : A polynomial formed from a function's derivatives at a chosen centre.
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■ Choose the mathematical structure

- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid. At $a=0$ it is a Maclaurin expansion. Finite sums may also simplify by cancellation or standard sum formulae.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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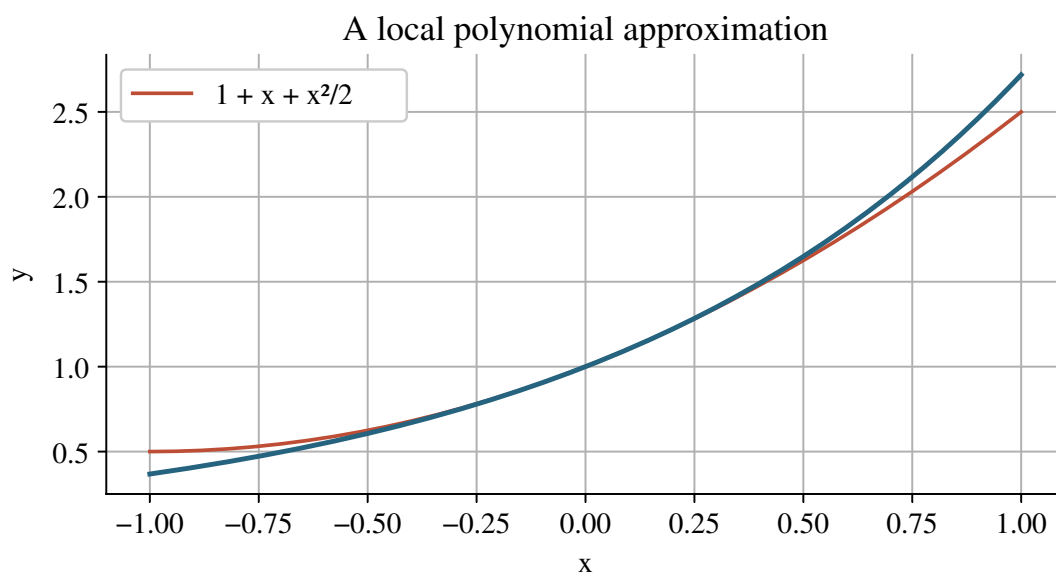
■ Work through a checked case

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Example:

The first three terms of e^x are $1+x+\frac{x^2}{2}$. At $x=0.1$ this gives 1.105, close to $e^{0.1}\approx 1.105170$. For the sum of k^2 from $k=1$ to 5, $\frac{n(n+1)(2n+1)}{6}$ gives $5\times 6\times 11/6=55$.



■ Test a tempting shortcut

- The factorial belongs in the denominator of every Taylor coefficient. The expansion centre is not always zero. An approximation close to its centre can be poor far away.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Warn:

A Taylor polynomial is always exact for the original function at every input. This claim is false. Explain which definition or assumption it violates.

■ Interpret a new situation

- For a telescoping series, write several terms and identify both surviving ends. For a Taylor approximation, state its order and compare with a known value or a remainder estimate where required.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

■ Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key:

A polynomial formed from a function's derivatives at a chosen centre. Choose the relationship, show the method, check its assumptions and interpret the result.

2 Practice

2.1 Use $1+x+x^2/2$ to estimate $e^{0.1}$. Show the relation, substitution and units. [2]

2.2 Find the sum of k^2 for $k=1$ to 5 . Show the relation, substitution and units. [2]

2.3 Find the coefficient of x^2 in the Maclaurin expansion of e^x . Show the relation, substitution and units. [2]

2.4 Decide whether this claim is true and justify: A Taylor polynomial is always exact for the original function at every input. [2]

2.5 Which description correctly defines Taylor polynomial? Choose one answer.

- **A** A polynomial formed from a function's derivatives at a chosen centre.
 - **B** A numerical answer that needs no assumptions or interpretation
 - **C** A rule that remains valid even when its required conditions fail
- [1]

3 Exam-style questions

3.1 Approximate $e^{0.2}$ with $1+x+x^2/2$. Give a remainder bound using $e^\xi x^3/6$ with $0 < \xi < 0.2$ and explain what the bound certifies. [4]

Solutions

2.1 The truncated polynomial is $1+0.1+0.1^2/2=1.105$.

2.2 Add the squared terms: $1+4+9+16+25=55$.

2.3 The second derivative of e^x at zero is 1; divide by $2!=2$ to get $1/2$.

2.4 False. The factorial belongs in the denominator of every Taylor coefficient. The expansion centre is not always zero. An approximation close to its centre can be poor far away.

2.5 A A polynomial formed from a function's derivatives at a chosen centre.

3.1 Approximation 1.22; $0 < R < e^{0.2}(0.2)^3/6 \approx 0.001629$. The quadratic underestimates and the exact value lies between 1.22 and 1.221629. The bound depends on the stated Taylor theorem conditions.