

Exercise walk-through

1.7

Series, finite differences and Taylor expansions

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches

1.7 Series, finite differences and Taylor expansions

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS Taylor polynomial

Objective

Series, finite differences and Taylor expansions. Supported HL concept preparation; original classroom practice, not an official IB paper or assessed learner work.

1 Key knowledge and worked examples

■ Can a polynomial replace an exponential nearby?

- Near a known point, a complicated function can be replaced by a polynomial approximation. The approximation needs a centre and a valid scale.
- This lesson studies **Taylor polynomial** : A polynomial formed from a function's derivatives at a chosen centre.

■ **Choose the mathematical structure**

- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\dots$ when the expansion is valid. At $a=0$ it is a Maclaurin expansion. Finite sums may also simplify by cancellation or standard sum formulae.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

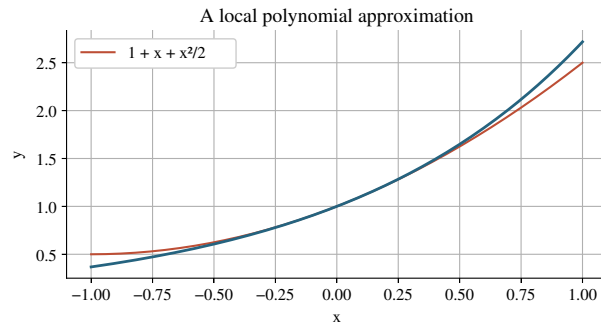
■ **Work through a checked case**

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Example:

The first three terms of e^x are $1+x+x^2/2$. At $x=0.1$ this gives 1.105, close to $e^{0.1} \approx 1.1051$.
For the sum of k^2 from $k=1$ to 5, $n(n+1)(2n+1)/6$ gives $5 \times 6 \times 11 / 6 = 55$.



■ **Test a tempting shortcut**

- The factorial belongs in the denominator of every Taylor coefficient. The expansion centre is not always zero. An approximation close to its centre can be poor far away.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Warn:

A Taylor polynomial is always exact for the original function at every input. This claim is false. Explain which definition or assumption it violates.

■ **Interpret a new situation**

- For a telescoping series, write several terms and identify both surviving ends. For a Taylor approximation, state its order and compare with a known value or a remainder estimate where required.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

■ **Use this in your course**

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key:

A polynomial formed from a function's derivatives at a chosen centre. Choose the relationship, show the method, check its assumptions and interpret the result.

Exercises 2.1-2.3

2.1 Use $1+x+x^2/2$ to estimate $e^{0.1}$. Show the relation, substitution and units. [2]

2.2 Find the sum of k^2 for $k=1$ to 5. Show the relation, substitution and units. [2]

2.3 Find the coefficient of x^2 in the Maclaurin expansion of e^x . Show the relation,

Exercises 2.1·2.3

2.1 Use $1+x+x^2/2$ to estimate $e^{0.1}$. Show the relation, substitution and units. [2]

2.1 The truncated polynomial is $1+0.1+0.1^2/2=1.105$.

2.2 Find the sum of k^2 for $k=1$ to 5. Show the relation, substitution and units. [2]

2.2 Add the squared terms: $1+4+9+16+25=55$.

2.3 Find the coefficient of x^2 in the Maclaurin expansion of e^x . Show the relation,

2.3 The second derivative of e^x at zero is 1; divide by $2!=2$ to get $1/2$.

Exercises 2.4·2.5

2.4 Decide whether this claim is true and justify: A Taylor polynomial is always exact for the original function at every input. [2]

2.5 Which description correctly defines Taylor polynomial? Choose one answer.

- **A** A polynomial formed from a function's derivatives at a chosen centre.
- **B** A numerical answer that needs no assumptions or interpretation
- **C** A rule that remains valid even when its required conditions fail [1]

3 Exam-style questions

Exercises 2.4-2.5

2.4 Decide whether this claim is true and justify: A Taylor polynomial is always exact for the original function at every input. [2]

2.4 False. The factorial belongs in the denominator of every Taylor coefficient. The expansion centre is not always zero. An approximation close to its centre can be poor far away.

2.5 Which description correctly defines Taylor polynomial? Choose one answer.

- **A** A polynomial formed from a function's derivatives at a chosen centre.
- **B** A numerical answer that needs no assumptions or interpretation
- **C** A rule that remains valid even when its required conditions fail [1]

3 Exam-style questions

2.5 A A polynomial formed from a function's derivatives at a chosen centre.

Exercise 3.1

3.1 Approximate $e^{0.2}$ with $1+x+x^2/2$. Give a remainder bound using $e^\xi x^3/6$ with $0 < \xi < 0.2$ and explain what the bound certifies. [4]

Exercise 3.1

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3.1 Approximation 1.22; $0 < R < e^{0.2}(0.2)^3/6 \approx 0.001629$. The quadratic underestimates and the exact value lies between 1.22 and 1.221629. The bound depends on the stated Taylor theorem conditions.