

Notes

International Baccalaureate · IB Diploma · Mathematics: Analysis
and Approaches · HL

AA: course teaching notes

Version: First assessment 2021; current through 2028. First-assessment-2029 course is separate.

These are original course-owned teaching notes. Objective-level exceptions are listed in the review, and lessons are tier labelled.

1 · Exact arithmetic and estimation

Can we pack without leftovers?

- A supplier packs 72 pencils and 90 pens into identical gift bags. How can we avoid leftovers?
 - This lesson studies **prime factor** : A prime number that divides the integer exactly.
-

Choose the mathematical structure

- Prime factors reveal shared structure. Use the smallest common prime powers for the HCF and the largest for the LCM. Estimate before calculating; use brackets to preserve the order of operations.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$a = \prod p_i^{\alpha_i}, \quad b = \prod p_i^{\beta_i}, \quad \text{HCF}(a, b) = \prod p_i^{\min(\alpha_i, \beta_i)}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$72=2^3 \times 3^2$ and $90=2 \times 3^2 \times 5$. Their HCF is $2 \times 9=18$. Make 18 bags with 4 pencils and 5 pens each. Their LCM is $2^3 \times 3^2 \times 5=360$.

$$72 = 2^3 \times 3^2$$

$$90 = 2 \times 3^2 \times 5$$

Shared smallest powers: $2 \times 3^2 = 18$

Test a tempting shortcut

- The HCF divides both numbers; the LCM is a multiple of both. They answer different questions. A decimal estimate is not an exact fraction.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The HCF of two positive integers is always larger than either integer. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For a non-calculator paper, keep fractions exact and show cancellation. For a calculator paper, enter the full expression and compare with your estimate.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A prime number that divides the integer exactly. Choose the relationship, show the method, check its assumptions and interpret the result.

1 · Indices, surds and standard form

How small is a microscopic length?

- A microscope records a length of 0.000072 metres. A compact representation must keep its size correct.
- This lesson studies **index** : The power to which a base is raised.

Choose the mathematical structure

- For the same positive base, multiplication adds indices and division subtracts them. A negative index means reciprocal; a fractional index represents a root. Standard form has $1 \leq a < 10$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

$$a^m a^n = a^{m+n}, \quad a^{-n} = \frac{1}{a^n}, \quad a^{m/n} = (\sqrt[n]{a})^m$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$0.000072 = 7.2 \times 10^{-5}$. Also $16^{(3/4)} = (16^{(1/4)})^3 = 2^3 = 8$. Simplify $\sqrt{72} = 6\sqrt{2}$, then rationalise $1/\sqrt{2} = \sqrt{2}/2$.

$$0.000072 = 7.2 \times 10^{-5}$$

7.2 is the coefficient; -5 is the power

The coefficient is between 1 and 10

Test a tempting shortcut

- Index laws do not turn a sum into a single power: $2^3 + 2^4 = 24$, not 2^7 . Do not round a surd when an exact answer is requested.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For every positive a , $a^2 + a^3$ equals a^5 . This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check powers of ten against the original quantity. Use surds for exact geometry, and round only the final length when the question asks for a decimal.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The power to which a base is raised. Choose the relationship, show the method, check its assumptions and interpret the result.

1 · Sequences, series and recurrence

Does the change add or multiply?

- A saving plan adds ¥30 more each week; a population model grows by 5% each year. Equal differences and equal ratios need different models.
 - This lesson studies **common ratio** : The constant multiplier between consecutive terms of a geometric sequence.
-

Choose the mathematical structure

- For an arithmetic progression, $u_n = a + (n-1)d$ and $S_n = n(2a + (n-1)d)/2$. For a geometric progression, $u_n = ar^{n-1}$ and $S_n = a(1-r^n)/(1-r)$. An infinite geometric sum exists only if $|r| < 1$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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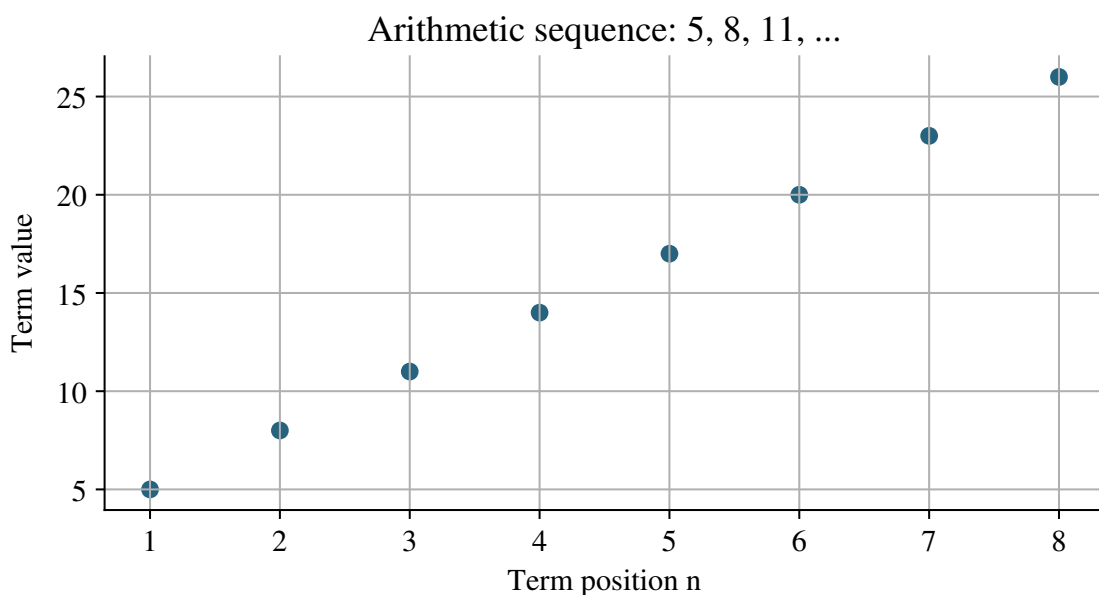
Work through a checked case

$$u_n = a + (n - 1)d, \quad S_n = \frac{n}{2} [2a + (n - 1)d]$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $a=5, d=3, n=8$, $u_8=5+7\times 3=26$ and $S_8=8(10+21)/2=124$. For $a=12, r=1/2$, $S_{\infty}=12/(1-1/2)=24$. For $u_{(n+1)}=2u_n+1$ with $u_1=1$, the next terms are 3, 7, 15.



Test a tempting shortcut

- The first term has index 1, so the exponent is $n-1$. A sequence is a list; a series is a sum. A geometric sequence can alternate in sign and still converge.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every geometric series has a finite sum to infinity. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Explain whether the context justifies additive or multiplicative change. In finance, distinguish a single deposit from a stream of deposits before choosing a sum formula.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The constant multiplier between consecutive terms of a geometric sequence. Choose the relationship, show the method, check its assumptions and interpret the result.

1 · Percentages, ratio and proportional reasoning

Can we recover the original price?

- A coat is reduced by 20% to ¥240. The discount applies to the original price, not to the sale price.
 - This lesson studies **multiplier** : A factor that performs a percentage change in one multiplication.
-

Choose the mathematical structure

- A $p\%$ increase has multiplier $1+p/100$; a decrease has multiplier $1-p/100$. Reverse a percentage by dividing by the multiplier. In a ratio, first find the total number of parts.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

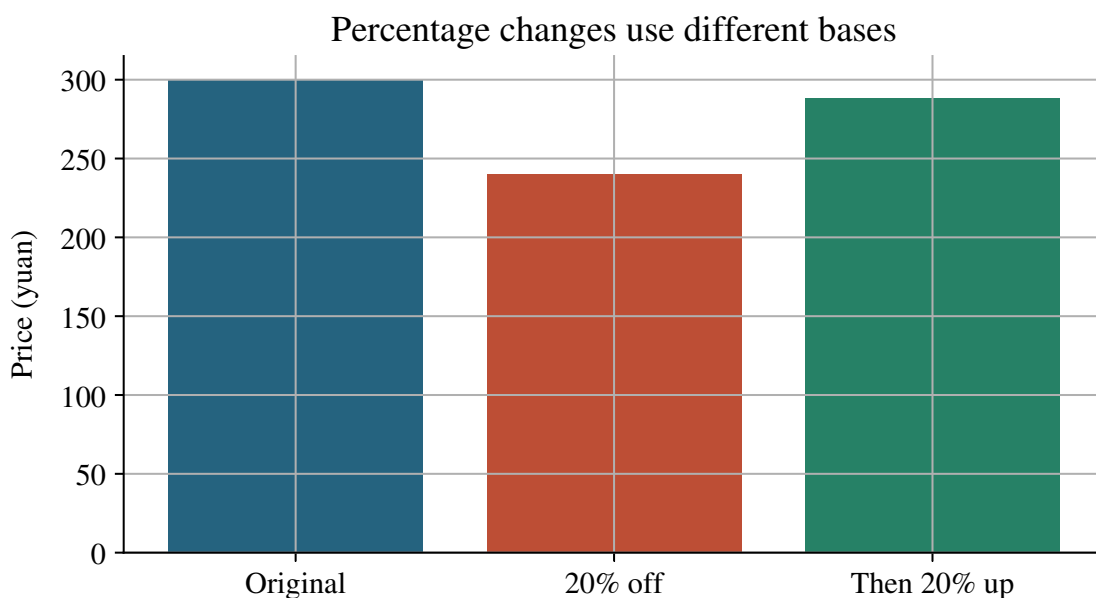
Work through a checked case

$$P_{\text{new}} = P_{\text{old}} \left(1 + \frac{r}{100} \right)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Let the original price be P . The model is sale price $= 0.8P$. Hence $P = 240/0.8 = 300$. A later 20% increase gives $240 \times 1.2 = 288$, so the two changes do not cancel.



Test a tempting shortcut

- A percentage uses a stated base. Subtracting the percentages loses that base. For compound change, multiply the multipliers; do not add the percentages.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A 20% decrease followed by a 20% increase restores the starting price. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For direct proportion use $y=kx$; for inverse proportion use $y=k/x$. Calculate k from a known pair before using a new value. State what you held constant.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

A factor that performs a percentage change in one multiplication. Choose the relationship, show the method, check its assumptions and interpret the result.

1 · Complex numbers and roots

What if the root is not real?

- The equation $x^2+1=0$ has no real roots. Extending the number system lets us represent and calculate with both roots.
 - This lesson studies **complex conjugate** : The number obtained by reversing the sign of the imaginary part.
-

Choose the mathematical structure

- Write $z=a+bi$ with $i^2=-1$. The conjugate is $a-bi$ and z times its conjugate $=a^2+b^2$. The modulus is $\sqrt{a^2+b^2}$; the argument needs the correct quadrant. Conjugate roots occur for polynomials with real coefficients.

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

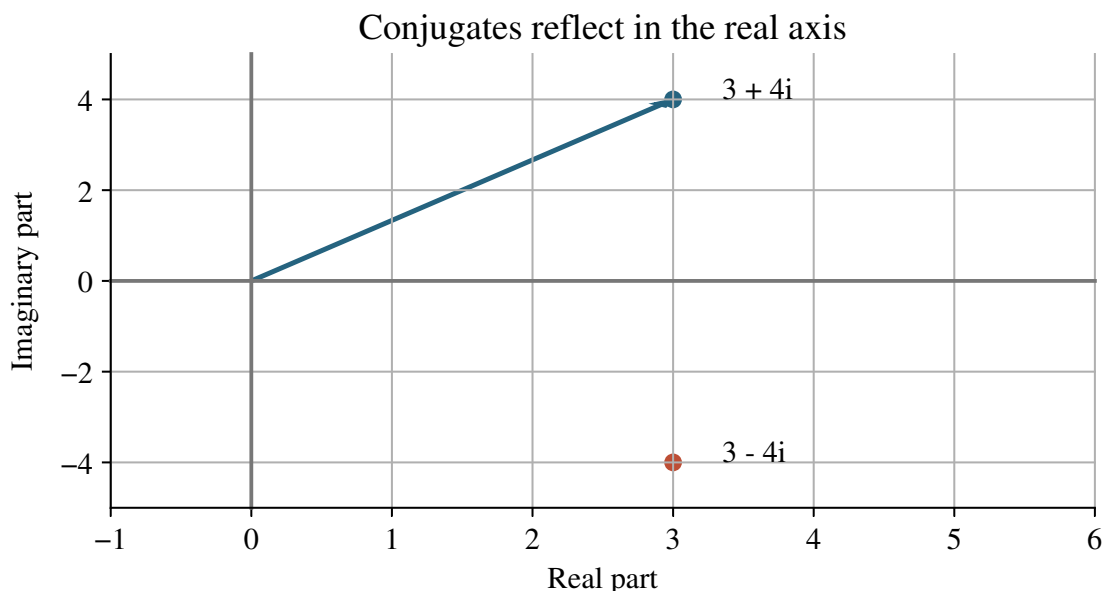
Work through a checked case

$$z = a + bi, \quad |z| = \sqrt{a^2 + b^2}, \quad z\bar{z} = a^2 + b^2$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $z=3+4i$, $|z|=5$ and z conjugate= $3-4i$. The reciprocal is $(3-4i)/25$. For $x^2-6x+13=0$, the roots are $3\pm 2i$. Multiplying by the conjugate makes a complex denominator real.



Test a tempting shortcut

- The square root of a sum is not the sum of the square roots. An inverse tangent alone may return the wrong quadrant. Do not confuse modulus with squared modulus.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For $a+bi$, its modulus is always $a+b$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Draw the real axis horizontally and imaginary axis vertically. For products, moduli multiply and arguments add; powers extend this geometric pattern through de Moivre's theorem.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The number obtained by reversing the sign of the imaginary part. Choose the relationship, show the method, check its assumptions and interpret the result.

1 · Binomial expansion and valid approximations

How does a small change affect a power?

- A small measurement change affects a power of a quantity. An expansion can show the size of first and second effects.
 - This lesson studies **binomial coefficient** : The number of ways to choose a specified number of objects from a set.
-

Choose the mathematical structure

- For positive integer n , $(a+b)^n$ is a finite binomial expansion. For noninteger n , expand $(1+x)^n$ as $1+nx+\frac{n(n-1)}{2}x^2+\dots$ with $|x|<1$. Factor out constants before using this form.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

$(1+2x)^5=1+10x+40x^2+80x^3+80x^4+32x^5$. For $(1+x)^{-1}$, the first three terms are $1-x+x^2$; at $x=0.1$ this gives 0.91 versus the exact $1/1.1\approx 0.909091$.

$$(1+2x)^5$$

Coefficients: 1, 10, 40, 80, 80, 32

The x^2 coefficient includes $(2x)^2$

Test a tempting shortcut

- An expansion in $2x$ requires $|2x| < 1$ for the infinite series, not merely $|x| < 1$. Positive-integer expansions are finite and do not have that convergence restriction.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every binomial expansion is a finite polynomial, including powers that are not positive integers. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- State the range of validity alongside an approximation. Retain enough terms to support the requested accuracy, and distinguish a coefficient from the whole term containing x .
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The number of ways to choose a specified number of objects from a set. Choose the relationship, show the method, check its assumptions and interpret the result.

1 · Series, finite differences and Taylor expansions

Can a polynomial replace an exponential nearby?

- Near a known point, a complicated function can be replaced by a polynomial approximation. The approximation needs a centre and a valid scale.

- This lesson studies **Taylor polynomial** : A polynomial formed from a function's derivatives at a chosen centre.

Choose the mathematical structure

- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid. At $a=0$ it is a Maclaurin expansion. Finite sums may also simplify by cancellation or standard sum formulae.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

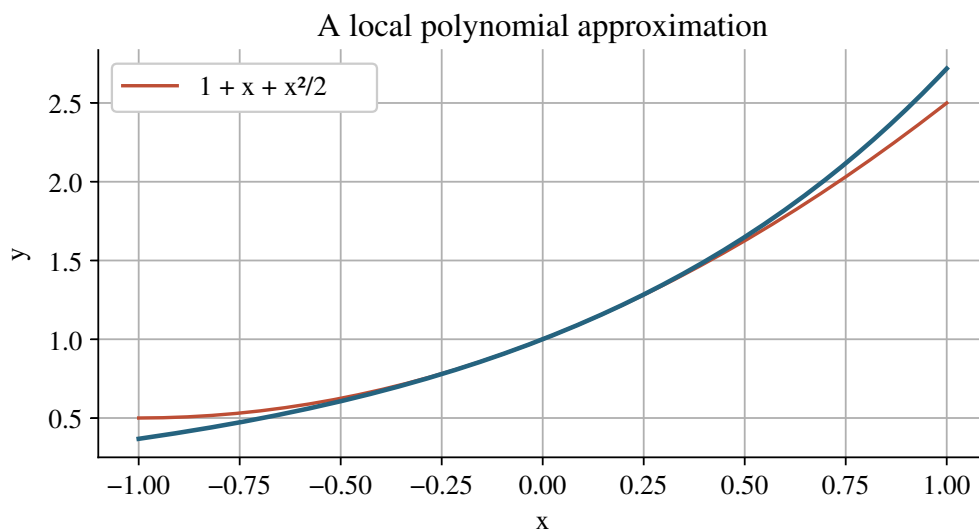
Work through a checked case

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The first three terms of e^x are $1+x+x^2/2$. At $x=0.1$ this gives 1.105, close to $e^{0.1} \approx 1.105170$. For the sum of k^2 from $k=1$ to 5, $n(n+1)(2n+1)/6$ gives $5 \times 6 \times 11 / 6 = 55$.



Test a tempting shortcut

- The factorial belongs in the denominator of every Taylor coefficient. The expansion centre is not always zero. An approximation close to its centre can be poor far away.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A Taylor polynomial is always exact for the original function at every input. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For a telescoping series, write several terms and identify both surviving ends. For a Taylor approximation, state its order and compare with a known value or a remainder estimate where required.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

A polynomial formed from a function's derivatives at a chosen centre. Choose the relationship, show the method, check its assumptions and interpret the result.

2 · Domains, inverses and composition

Which inputs are allowed?

- A square-root model returns a real output only for some inputs. Its formula alone does not specify a complete function.
 - This lesson studies **domain** : The set of allowed inputs to a function.
-

Choose the mathematical structure

- State the domain and range. For an inverse, first ensure the function is one-to-one on its domain. Composition fg means apply g first, then f ; the intermediate output must be an allowed input to f .
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

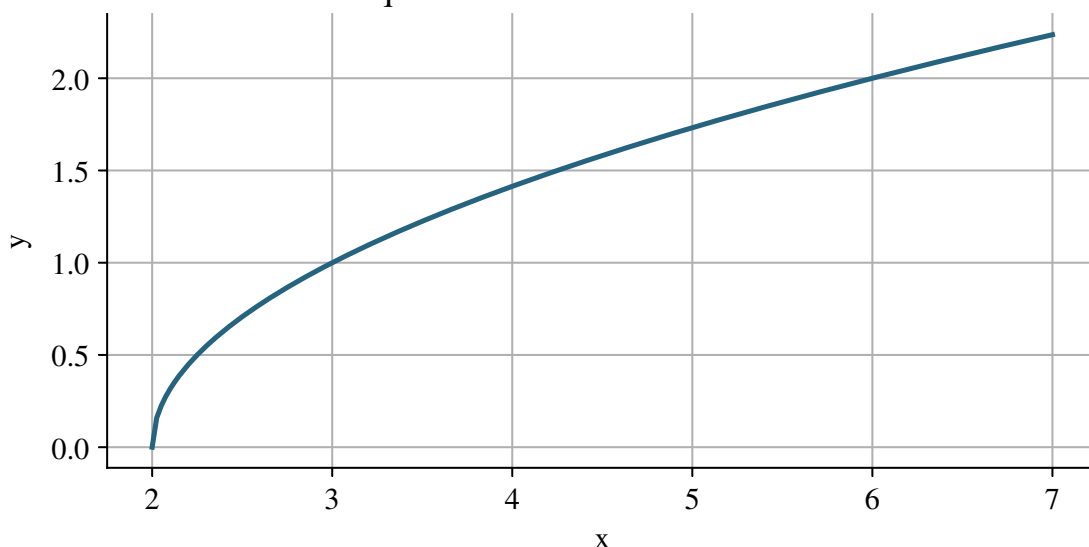
$$f(g(x)) = (f \circ g)(x), \quad f^{-1}(f(x)) = x$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $f(x)=\sqrt{x-2}$, $x \geq 2$ and the range is $y \geq 0$. From $y=\sqrt{x-2}$, $x=y^2+2$. Thus f inverse(x)= x^2+2 with $x \geq 0$. For $g(x)=x+3$, $fg(1)=f(4)=\sqrt{2}$.

Square-root domain starts at $x = 2$



Test a tempting shortcut

- Squaring can introduce extraneous solutions. Restricting a parabola's domain is essential before claiming an inverse. A horizontal translation inside f has the opposite sign to the graph's movement.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every quadratic function on all real numbers has an inverse function. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check $f(f^{-1}(x))=x$ on the inverse domain. Use a sketch to test whether a horizontal line meets the original graph more than once.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The set of allowed inputs to a function. Choose the relationship, show the method, check its assumptions and interpret the result.

2 · Exponentials, logarithms and modelling

Why does a decay model stay positive?

- A medicine concentration falls by the same percentage each hour. A constant subtraction would eventually predict a negative amount.
- This lesson studies **half-life** : The time for a decaying quantity to fall to half its initial value.

Choose the mathematical structure

- For $y = Ae^{kt}$, k is a proportional rate. Taking logs gives $\ln y = \ln A + kt$. Logarithms require positive arguments, and $\log(x+y)$ is not $\log x + \log y$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

Work through a checked case

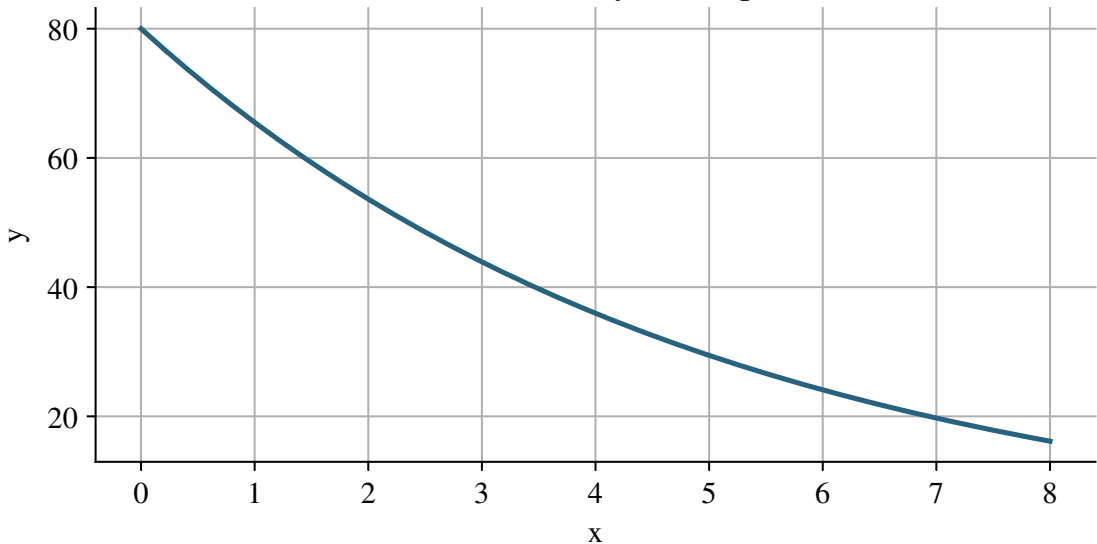
$$y = Ae^{kt}, \quad \ln y = \ln A + kt$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

If $y = 80e^{(-0.2t)}$, $y = 40$ gives $e^{(-0.2t)} = 1/2$. Hence $t = \ln 2 / 0.2 \approx 3.466$. For $3^x = 20$, $x = \ln 20 / \ln 3$. Plotting $\ln y$ against t linearises this exponential model.

Same fractional decay over equal times



Test a tempting shortcut

- A fitted exponential is a model, not a guarantee. Specify the time units and range of use. A negative k describes decay; a negative starting amount usually contradicts the context.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

For positive x and y , $\ln(x+y) = \ln x + \ln y$. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Compare actual observations with the model. Systematic departures may indicate changing conditions. In a report, explain what the rate and initial value mean, rather than giving a bare equation.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The time for a decaying quantity to fall to half its initial value. Choose the relationship, show the method, check its assumptions and interpret the result.

3 · Angle reasoning, similarity and mensuration

Does volume scale like length?

- A model has lengths one third of the real object. How much smaller are its area and volume?
 - This lesson studies **scale factor** : The multiplier that relates corresponding lengths in similar shapes.
-

Choose the mathematical structure

- For similar shapes with length scale factor k , areas scale by k^2 and volumes by k^3 . State angle reasons explicitly. A circle's tangent is perpendicular to the radius at the contact point.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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Work through a checked case

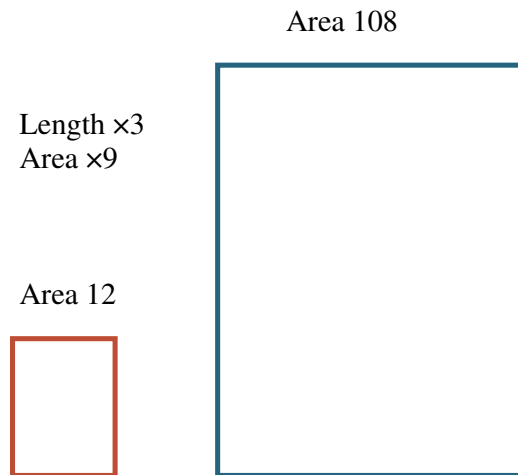
$$\frac{A_2}{A_1} = k^2, \quad \frac{V_2}{V_1} = k^3$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

If model-to-real length factor is 3, a model area of 12 cm^2 gives $12 \times 3^2 = 108 \text{ cm}^2$ and a model volume of 8 cm^3 gives $8 \times 3^3 = 216 \text{ cm}^3$. A cylinder with $r=3, h=5$ has volume $\pi r^2 h = 45\pi$.

A length scale factor is squared for area



Test a tempting shortcut

- Equal angles alone establish similarity, not equal size. Use corresponding lengths in the same order. Convert linear units before calculating area or volume, or square/cube the conversion factor correctly.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Doubling every length of a solid doubles its volume. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- A geometric proof should name the relevant theorem, identify the equal angle or ratio, and draw the conclusion. A scale drawing is evidence only when the task permits measurement.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The multiplier that relates corresponding lengths in similar shapes. Choose the relationship, show the method, check its assumptions and interpret the result.

3 · Right triangles and non-right triangles

Which side does the ladder need?

- A ladder reaches a height of 4 m while its foot is 3 m from a wall. Which sides are known, and which angle do we need?
 - This lesson studies **hypotenuse** : The side opposite the right angle in a right-angled triangle.
-

Choose the mathematical structure

- In a right triangle $a^2 + b^2 = c^2$; $\sin\theta = \text{opposite/hypotenuse}$, $\cos\theta = \text{adjacent/hypotenuse}$ and $\tan\theta = \text{opposite/adjacent}$. For other triangles, use the sine or cosine rule, or $\text{area} = ab \sin C/2$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

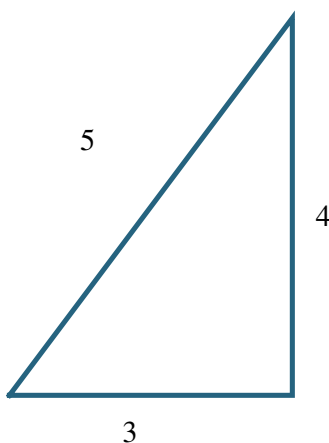
$$a^2 + b^2 = c^2, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

The ladder length is $c = \sqrt{3^2 + 4^2} = 5$ m. Its angle to the ground satisfies $\tan \theta = 4/3$, so $\theta \approx 53.1^\circ$. With two sides 6 and 8 enclosing 60° , $c^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \cos 60^\circ = 52$.

Label the known sides before choosing a rule



Test a tempting shortcut

- Label sides relative to the chosen angle. Pythagoras needs a right angle. A calculator angle mode error can produce a plausible but wrong result. Keep unrounded values for later steps.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Pythagoras applies to every triangle, including triangles without a right angle. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Use a plan or elevation for a three-dimensional problem before applying a triangle rule. Explain why the chosen triangle contains the required length or angle.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

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Key idea

The side opposite the right angle in a right-angled triangle. Choose the relationship, show the method, check its assumptions and interpret the result.

3 · Radians, identities and trigonometric equations

How far does the rim travel?

- A rotating wheel sweeps an arc. Radians let its angle connect directly to the distance travelled along the rim.
 - This lesson studies **radian** : The angle subtended by an arc whose length equals the radius.
-

Choose the mathematical structure

- For θ in radians, arc length $s=r\theta$ and sector area $A=\frac{1}{2}r^2\theta$. Use $\sin^2\theta+\cos^2\theta=1$ and angle identities to simplify or solve. List every solution in the stated interval.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

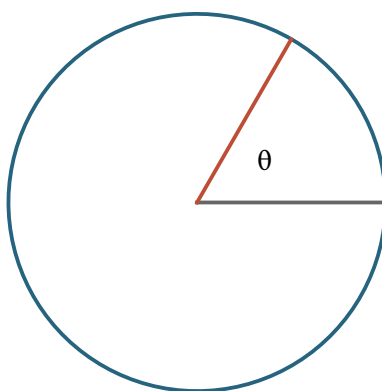
$$s = r\theta, \quad A = \frac{1}{2}r^2\theta, \quad \sin^2 \theta + \cos^2 \theta = 1$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $r=4$ and $\theta=\pi/3$, $s=4\pi/3$ and $A=8\pi/3$. Solving $\sin x=1/2$ on $0 \leq x < 2\pi$ gives $x=\pi/6$ and $5\pi/6$. For $\sin 2x=1/2$, solve first on $0 \leq 2x < 4\pi$ before dividing by 2.

A radius fixes distance from the centre



radius $\times$ angle gives arc length

Test a tempting shortcut

- A calculator's inverse sine gives one principal value. It does not give every solution. Dividing by $\sin x$ can discard solutions where $\sin x=0$; check that case first.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The principal inverse sine value is always the only solution in a full turn. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose an identity that reduces the equation to one trigonometric function. Keep exact familiar angles where possible, and check each solution against the original equation.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The angle subtended by an arc whose length equals the radius. Choose the relationship, show the method, check its assumptions and interpret the result.

3 · Vectors and transformation geometry

Why is displacement shorter than the walk?

- Walking 4 m east and 3 m north gives a displacement of 5 m, even though the travelled distance is 7 m.
 - This lesson studies **resultant** : The vector sum representing the combined displacement or force.
-

Choose the mathematical structure

- Add corresponding vector components. A displacement from A to B is $\vec{OB}-\vec{OA}$. Parallel vectors are scalar multiples. A vector has magnitude and direction; a distance is a scalar.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

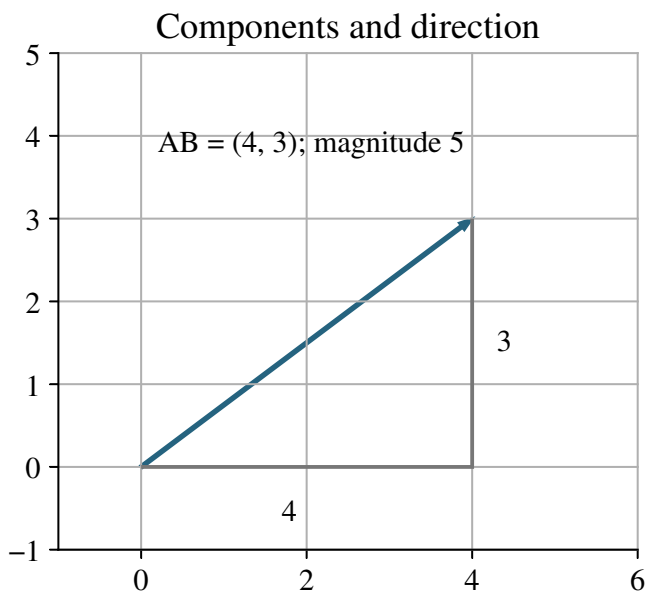
Work through a checked case

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad |\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

With $\mathbf{a}=(4,1)$ and $\mathbf{b}=(1,3)$, $\mathbf{a}+\mathbf{b}=(5,4)$. Their scalar product is $\mathbf{a} \cdot \mathbf{b}=4 \times 1+1 \times 3=7$. The magnitude of $\mathbf{a}$ is $\sqrt{17}$. For $\mathbf{A}=(1,2)$, $\mathbf{B}=(5,5)$, $\overrightarrow{AB}=(4,3)$ and $|\overrightarrow{AB}|=5$.



Test a tempting shortcut

- The order of subtraction matters: $\mathbf{BA} = -\mathbf{AB}$. Proving parallelism needs a scalar-multiple relation; a sketch alone is insufficient. Negative enlargement reverses position about its centre.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

$\overrightarrow{AB}$ and $\overrightarrow{BA}$ always have the same components. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For a translation, every point uses the same displacement. For advanced line problems, distinguish the position vector from a direction vector and solve parameter equations consistently.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The vector sum representing the combined displacement or force. Choose the relationship, show the method, check its assumptions and interpret the result.

4 · Data summaries, histograms and interpretation

Can one average tell the whole story?

- Two groups have the same median but different spread. One summary cannot describe both location and consistency.
 - This lesson studies **frequency density** : Frequency divided by class width, used as histogram height.
-

Choose the mathematical structure

- Compare an appropriate average and spread in context. A histogram uses area for frequency, so $\text{height} = \text{frequency} / \text{class width}$. Grouped estimates assume representative values within intervals.

- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

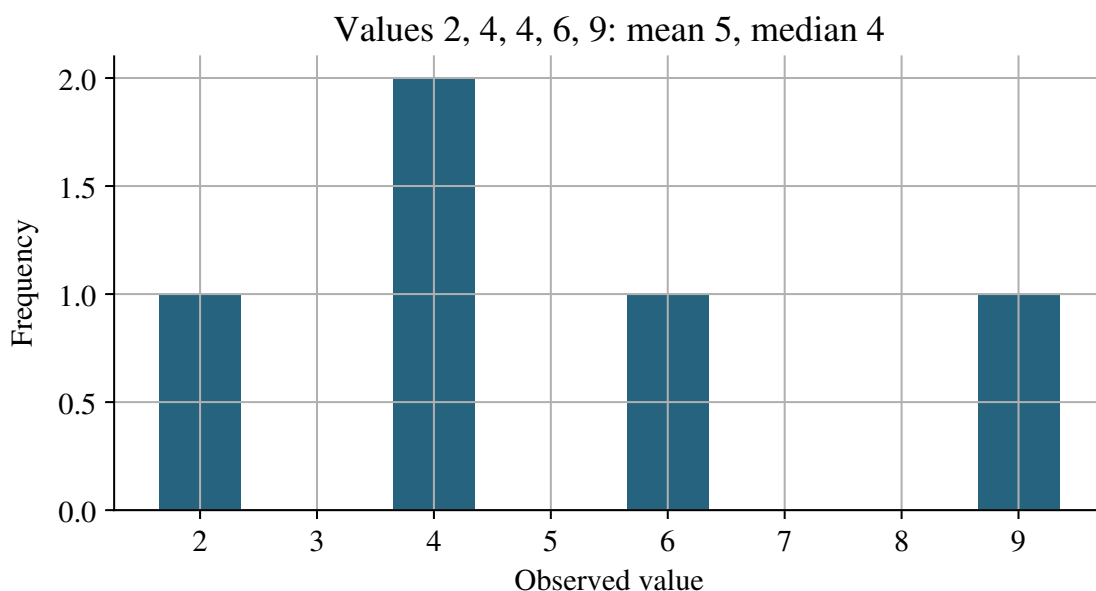
Work through a checked case

$$\bar{x} = \frac{\sum x_i}{n}, \quad \text{density} = \frac{\text{frequency}}{\text{class width}}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A class from 10 to 20 with frequency 30 has density $30/10=3$. A class from 20 to 40 with frequency 20 has density $20/20=1$. Its wider bar must not be mistaken for a larger density. For values 2,4,4,6,9, the median is 4 and mean is 5.



Test a tempting shortcut

- The tallest histogram bar need not contain the most observations. A grouped mean is an estimate. Correlation does not prove causation, and extrapolation extends beyond the observed range.

- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A histogram bar's height always equals its frequency, even with unequal class widths. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose a display that fits the data type. Give both a numerical comparison and what it means for the population; do not infer more precision than the sample supports.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

Frequency divided by class width, used as histogram height. Choose the relationship, show the method, check its assumptions and interpret the result.

4 · Probability, trees and conditional reasoning

What changes after the first draw?

- A bag has 3 red and 2 blue counters. Taking two without replacement changes the chance of the second colour.
 - This lesson studies **conditional probability** : The probability of an event after restricting the sample space to a stated condition.
-

Choose the mathematical structure

- Multiply along a tree branch and add disjoint branches. With replacement, the composition stays fixed. Conditional probability is $P(A \text{ given } B) = P(A \cap B) / P(B)$, for $P(B) > 0$.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

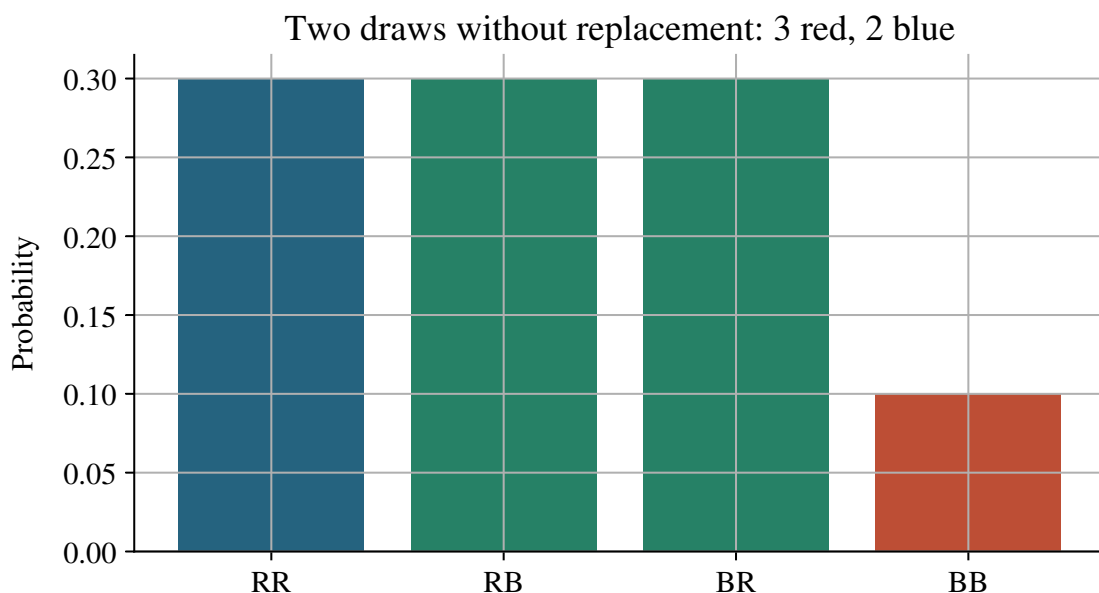
Work through a checked case

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

Without replacement, $P(\text{two red}) = 3/5 \times 2/4 = 3/10$. $P(\text{one of each}) = 3/5 \times 2/4 + 2/5 \times 3/4 = 3/5$. If $P(A \cap B) = 0.12$ and $P(B) = 0.3$, $P(A \text{ given } B) = 0.4$.



Test a tempting shortcut

- Mutually exclusive means no overlap; independent means that knowing one event does not change the other's probability. Two disjoint events with positive probability are not independent.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Mutually exclusive events with positive probabilities must be independent. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- A two-way table makes the restricted denominator visible. Before using a product $P(A)P(B)$, justify independence from the context or the supplied information.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The probability of an event after restricting the sample space to a stated condition. Choose the relationship, show the method, check its assumptions and interpret the result.

4 · Binomial, normal and Poisson models

What makes a count predictable?

- A quality inspector counts defective items. The number is random, but a model can describe its likely range.

- This lesson studies **expected value** : The probability-weighted mean of a random variable.

Choose the mathematical structure

- A binomial model needs fixed n , independent trials, two outcomes and constant p . $E(X)=np$ and $\text{Var}(X)=np(1-p)$. For a normal model use $z=(x-\mu)/\sigma$. A Poisson model describes counts with a constant rate and appropriate independence assumptions.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

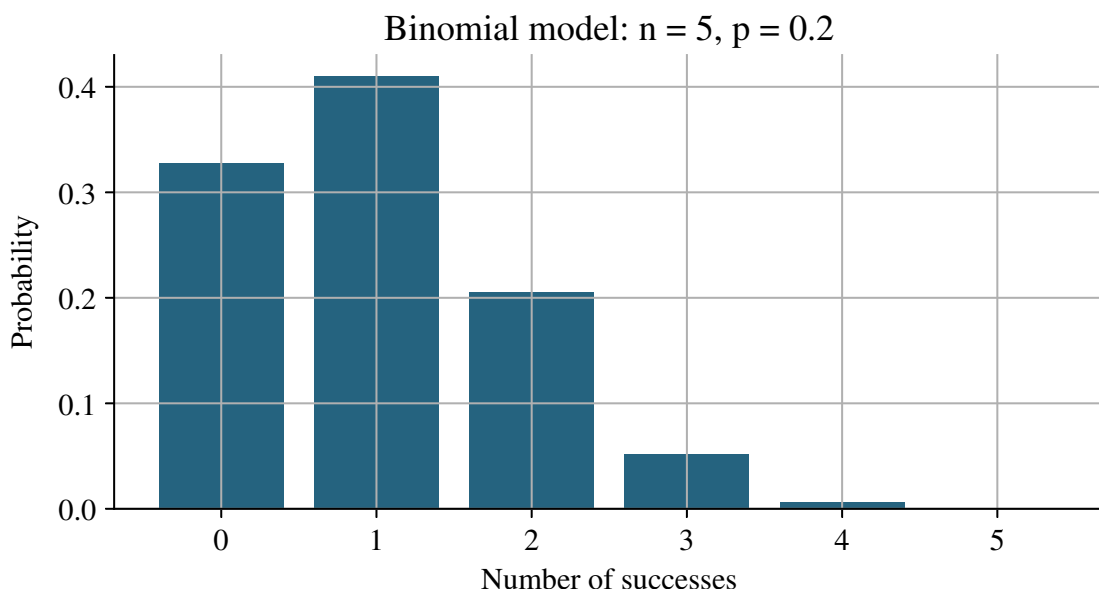
Work through a checked case

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad X \sim B(n, p)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For X binomial(5,0.2), $P(X=0)=0.8^5=0.32768$, $E(X)=1$ and $\text{Var}(X)=0.8$. For a normal quantity with $\mu=100, \sigma=15$, the value 130 has $z=2$. A Poisson mean of 3 per hour gives mean 6 over two hours.



Test a tempting shortcut

- Not every count is binomial: changing p or dependence can invalidate it. For a continuous variable, the probability of one exact value is zero. Continuity correction matters when approximating a discrete distribution by a normal one.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every count of successes has a binomial distribution regardless of dependence. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Write the event as an inequality before using calculator distribution functions. Distinguish $P(X < k)$, $P(X \leq k)$ and a tail complement. State assumptions in context.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The probability-weighted mean of a random variable. Choose the relationship, show the method, check its assumptions and interpret the result.

5 · Derivatives and stationary points

What is the slope at one point?

- A curved road has different slopes at different positions. An average gradient cannot describe every point.
 - This lesson studies **derivative** : The instantaneous rate of change, also the gradient of a tangent.
-

Choose the mathematical structure

- For $y = ax^n$, $dy/dx = anx^{n-1}$. A stationary point satisfies $dy/dx = 0$. Check the sign change of the derivative, or the second derivative when it is nonzero, to classify it.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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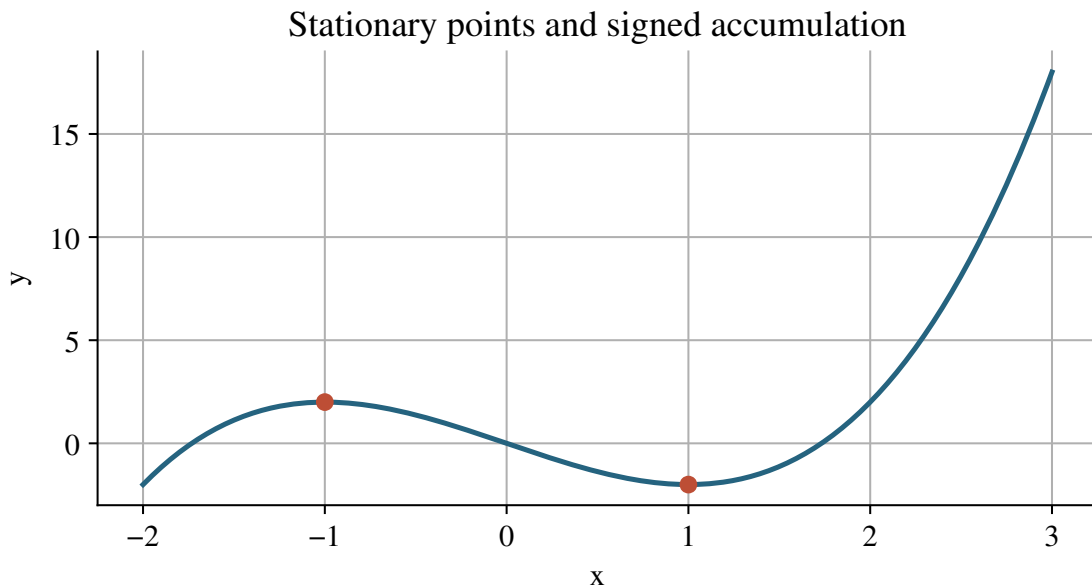
Work through a checked case

$$\frac{d}{dx}(ax^n) = anx^{n-1}, \quad f'(x) = 0$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y=x^3-3x$, $dy/dx=3x^2-3$. At $x=1$, the gradient is 0 and $y=-2$. The second derivative is $6x$, positive at $x=1$, so this is a local minimum. At $x=-1$, $y=2$ and the second derivative is negative, giving a local maximum.



Test a tempting shortcut

- A zero derivative does not always mean a maximum or minimum: $y=x^3$ is stationary at 0 but continues increasing. An endpoint can also produce an extreme value on a restricted domain.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every point with zero derivative is a local maximum or minimum. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- At GCSE/IGCSE use only the polynomial scope allowed by the tier; do not add chain, product or quotient rules there. At advanced level, connect the derivative to rates and optimization with a valid domain.

- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The instantaneous rate of change, also the gradient of a tangent. Choose the relationship, show the method, check its assumptions and interpret the result.

5 · Integrals, area and accumulation

How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?
 - This lesson studies **antiderivative** : A function whose derivative equals the given integrand.
-

Choose the mathematical structure

- For $n \neq -1$, the integral of ax^n is $ax^{n+1}/(n+1) + C$. A definite integral is signed accumulation. Split at sign changes when total area or distance is required.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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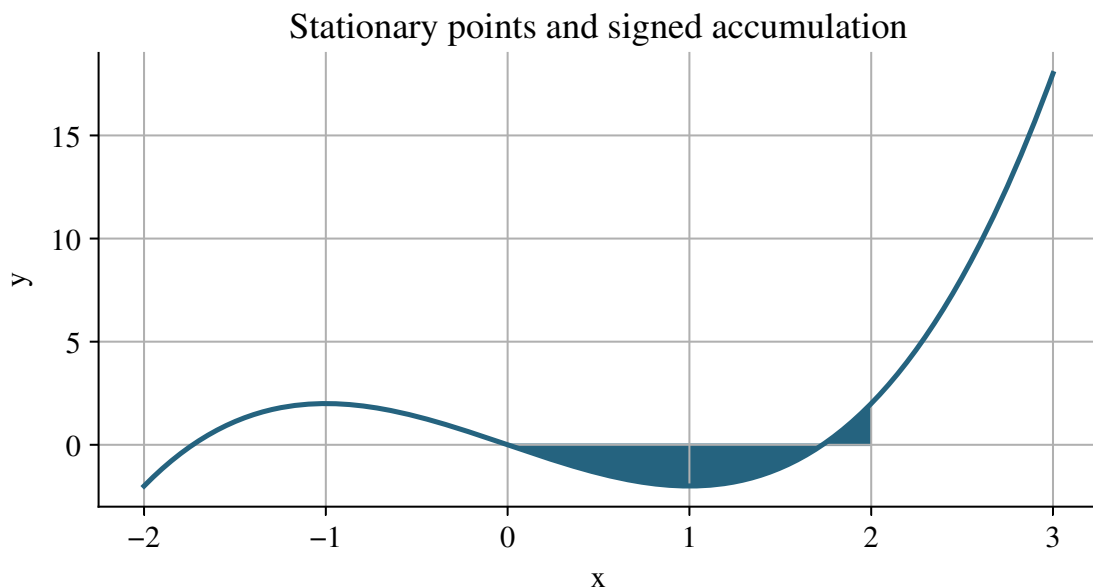
Work through a checked case

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $v(t) = 3t^2 - 3$ over $0 \leq t \leq 2$, displacement $= [t^3 - 3t]_0^2 = 2$. Since v changes sign at $t = 1$, distance $= -[t^3 - 3t]_0^1 + [t^3 - 3t]_1^2 = 2 + 4 = 6$. The constants cancel only for a definite integral.



Test a tempting shortcut

- The integral of $1/x$ is $\ln|x| + C$, not the power rule with $n = -1$. Signed area can be zero even when the total enclosed area is positive.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

A definite integral always equals the total positive area, even when the graph crosses the axis. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Identify what the integral means and include the correct units. A rate measured per second integrates to the underlying quantity, not to another rate. Differentiate an antiderivative to check it.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A function whose derivative equals the given integrand. Choose the relationship, show the method, check its assumptions and interpret the result.

5 · Chain, product, quotient and implicit differentiation

Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
 - This lesson studies **chain rule** : The rule that multiplies the outer derivative by the inner derivative for a composite function.
-

Choose the mathematical structure

- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$. For uv , differentiate to $u'v + uv'$. For u/v , use $(u'v - uv')/v^2$, where $v \neq 0$.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

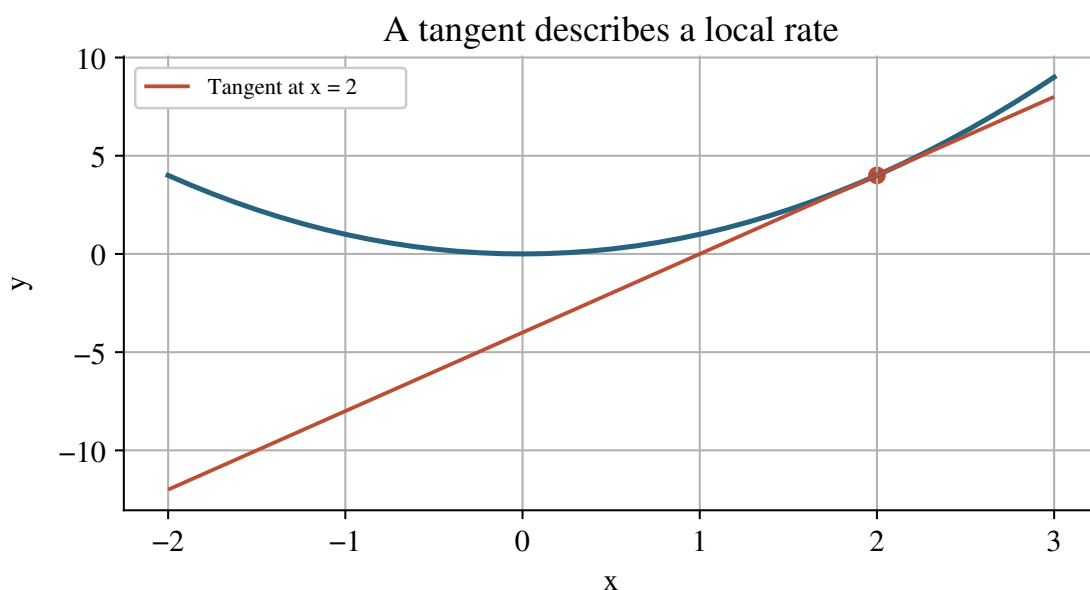
Work through a checked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For $y=(3x+1)^4$, $y' = 4(3x+1)^3 \times 3 = 12(3x+1)^3$. At $x=0$, the gradient is 12.
For $x^2+y^2=25$, differentiate implicitly: $2x+2y y' = 0$, so $y' = -x/y$ when $y \neq 0$.



Test a tempting shortcut

- A derivative of a product is not the product of the derivatives. In implicit differentiation, every differentiated function of y brings a dy/dx factor. A quotient's denominator is squared.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The derivative of $u(x)v(x)$ is always u prime times v prime. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Choose a useful form before differentiating: expanding a short polynomial may be simpler. For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

The rule that multiplies the outer derivative by the inner derivative for a composite function. Choose the relationship, show the method, check its assumptions and interpret the result.

5 · Substitution, parts and partial fractions

Can we integrate the two factors separately?

- A simple-looking product such as $x e^x$ cannot be integrated by separately integrating each factor.
 - This lesson studies **integration by parts** : An integration method based on the derivative of a product.
-

Choose the mathematical structure

- Use substitution when an inner derivative appears as a factor. By parts, $\int u v' = uv - \int u' v$. For a rational function, divide first if needed, then decompose into partial fractions.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
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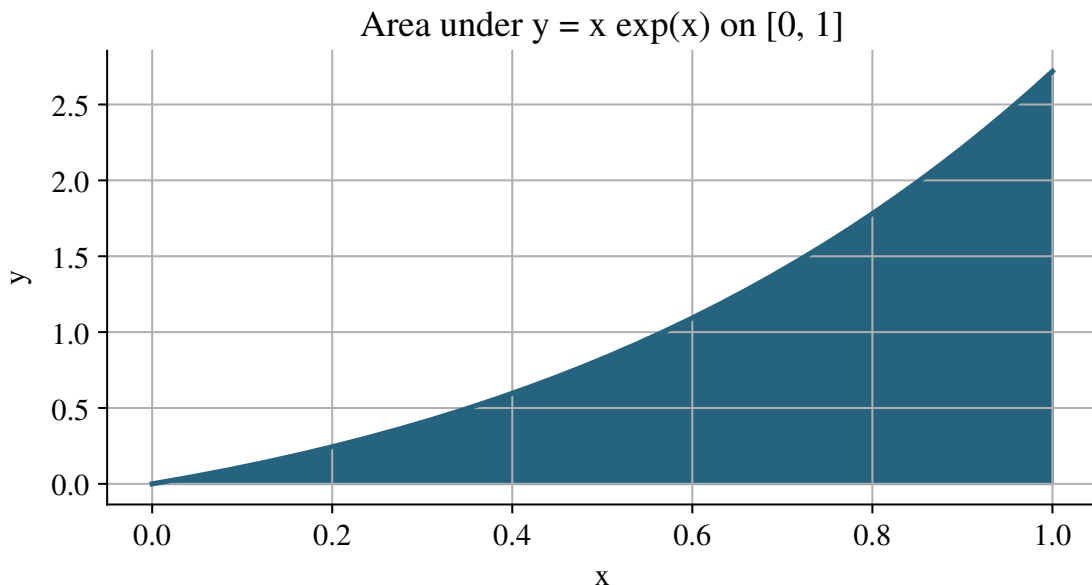
Work through a checked case

$$\int uv' dx = uv - \int u'v dx$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

For integral $\int x e^x dx$, take $u=x$ and $v' = e^x$. Then the integral is $x e^x - e^x + C$. For integral $\int \frac{2x}{x^2+1} dx$, substitute $w=x^2+1$, $dw=2x dx$; the answer is $\ln(x^2+1) + C$.



Test a tempting shortcut

- Choosing u and v prime well matters: the remaining integral should become simpler. In a definite substitution, either change the limits or return to x before applying the original limits.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The integral of a product equals the product of its separate integrals. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- Check by differentiating. For volumes of revolution around the x -axis use $V = \pi \int y^2 dx$; do not confuse the square of a function with the integral of the function.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

An integration method based on the derivative of a product. Choose the relationship, show the method, check its assumptions and interpret the result.

5 · Differential equations and numerical solutions

What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.

- This lesson studies **initial condition** : A specified value that selects a particular solution from a family.

Choose the mathematical structure

- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$. Use an initial condition to find A. Euler's method takes $y_{next}=y+h f(x,y)$, with a chosen step h .
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

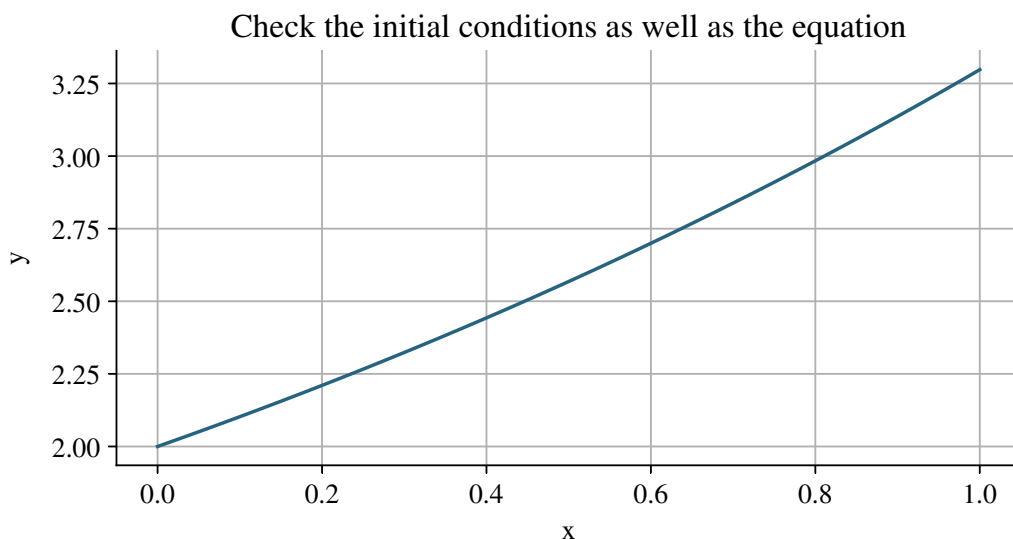
Work through a checked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

With $dy/dx=0.5y$ and $y(0)=2$, $y=2e^{(0.5x)}$. Euler with $h=0.2$ gives $y(0.2)\approx 2+0.2\times 1=2.2$ and $y(0.4)\approx 2.2+0.2\times 1.1=2.42$. The exact second value is about 2.4428.



Test a tempting shortcut

- An initial condition determines the integration constant; it is not a replacement for integrating. Dividing by y can omit an equilibrium solution $y=0$. Euler accuracy depends on step size and the equation.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

Every step size gives exactly the same Euler approximation. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions. This is Further Pure scope; check the named unit.
 - A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.
-

Use this in your course

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- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A specified value that selects a particular solution from a family. Choose the relationship, show the method, check its assumptions and interpret the result.

IA · Planning a mathematical exploration

What makes an investigation mathematical?

- A student wants to predict queue waiting time at the school canteen. A useful exploration needs a question, data and mathematics the student can explain.
 - This lesson studies **assumption** : A condition adopted to make a mathematical model possible and whose effect should be evaluated.
-

Choose the mathematical structure

- Frame a focused question with measurable variables and a feasible method. Keep a dated record of decisions, sources and mathematical reasoning. Use technology to support analysis, and explain every important output in your own words.
 - State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.
-

Work through a checked case

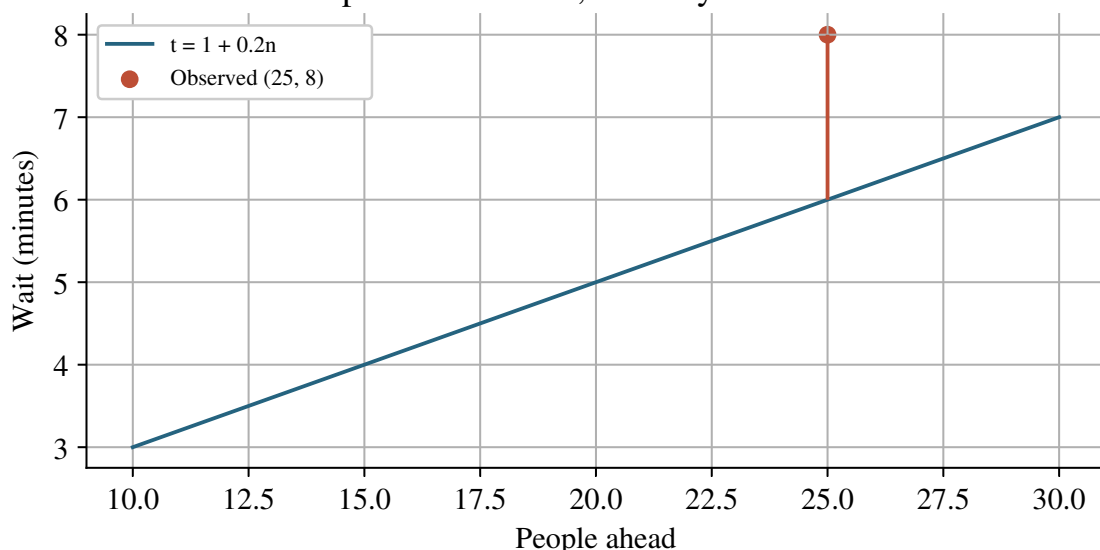
$$\hat{t} = 1 + 0.2n, \quad e = t - \hat{t}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Worked example

A pilot records mean waiting times of 3,5,7 minutes when 10,20,30 people are ahead. A first model is $t=1+0.2n$. At $n=25$ it predicts 6 minutes. An actual wait of 8 gives residual 2; investigate varying service rates rather than hiding the mismatch.

Interpret the residual, not only the fitted line



Test a tempting shortcut

- Do not copy a sample exploration, invent data or manufacture personal engagement. A prediction is not evidence until it is compared with observations. Consent and privacy matter when collecting school data.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Watch out

The school-supervised mathematical exploration can be replaced by a timed written mock exam. This claim is false. Explain which definition or assumption it violates.

Interpret a new situation

- The school supervises topic approval, checkpoints, authenticity and assessment. For the current 2021 course use the current school's guide and criteria; the 2029 inquiry criteria belong to a different assessment version.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key idea

A condition adopted to make a mathematical model possible and whose effect should be evaluated. Choose the relationship, show the method, check its assumptions and interpret the result.