

Coordinate operators, separated modes and stable updates

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Writing the Laplacian of a radial function as $\nabla^2 f$ drops a term; the operator belongs to the coordinates, not the symbol.

Prerequisites: Algebra, differentiation and vector components.

- Use the divergence and Laplacian in the coordinate system selected by the symmetry
- Solve a simple separated boundary-value mode problem
- Check convergence, units and stability of an original numerical update

Match operator to symmetry

Choose coordinates by symmetry before writing operators. For a radial function in spherical coordinates, $\nabla^2 f = f'' + (2/r)f'$, and the divergence of a radial field $A = A_r \hat{r}$ is $(1/r^2)d(r^2 A_r)/dr$. Check: $A = k \hat{r}/r^2$ has divergence $(1/r^2)dk/dr = 0$ away from the origin, and $f = 1/r$ satisfies $\nabla^2 f = f'' + 2f'/r = 2/r^3 - 2/r^3 = 0$ away from the origin. Cartesian forms apply only to Cartesian dependences.

Separate the modes

Separation of variables turns a boundary-value problem into ordinary modes. For a string fixed at both ends, write $u = X(x)T(t)$; the wave equation gives $X''/X = (1/c^2)T''/T = -k^2$, the fixed ends force $X = \sin(n\pi x/L)$, and the allowed frequencies are $\omega_n = n\pi c/L$, $f_n = nc/(2L)$. Each mode must satisfy the boundary conditions before any sum over modes.

Bound the step

An explicit Euler update for $y' = -\lambda y$ is $y_{n+1} = (1 - \lambda h)y_n$, bounded under $|1 - \lambda h| \leq 1$, i.e. $0 < h \leq 2/\lambda$; decay to zero needs the strict upper inequality. With $\lambda = 10/s$, h must not exceed 0.2 s; $h = 0.25$ s multiplies by -1.5 each step and diverges with alternating sign. Stability is a property of the recurrence, not of the true solution.

Verify the answer

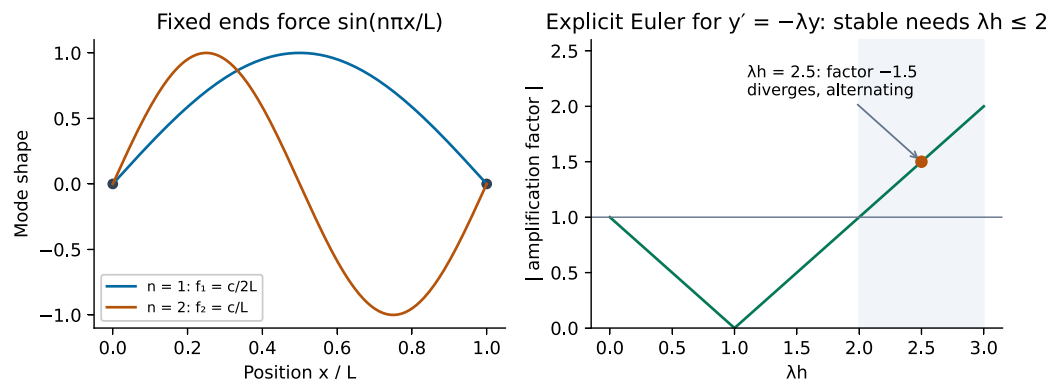
Check a numerical answer three ways before trusting it: units of every term (λh must be dimensionless), convergence under refinement (halve h and compare; explicit Euler error is first order), and a limiting case with a known exact answer. For $y' = -4y$ with $y_0 = 8$ and $h = 0.2$, five steps give $8 \times 0.2^5 = 0.00256$ versus exact $8e^{-4} \approx 0.1465$: stable but inaccurate, so refine.

Worked method

A spherical radial field has divergence

$$\nabla \cdot (A_r \hat{r}) = r^{-2} \frac{d}{dr}(r^2 A_r).$$

For $A_r = Cr$, it gives $3C$, not C . Fixed string ends require $X(0) = X(L) = 0$, hence $X_n = \sin(n\pi x/L)$ with positive integer n . Euler for $\dot{y} = -\lambda y$ gives $y_{n+1} = (1 - \lambda h)y_n$. For positive h , bounded absolute stability permits $h \leq 2/\lambda$; decay to zero requires the strict inequality $0 < h < 2/\lambda$. At the upper endpoint the numerical solution alternates without decay. Stability does not establish accuracy.



Coordinate operators, separated modes and stable updates: original GRE teaching diagram.

Check conditions and vocabulary

Applying the Cartesian Laplacian to a radial function, summing modes that violate the boundary conditions, or calling a bounded-looking run converged without refining the step. Match operator, boundaries and stability condition to the actual problem.

separation of variables: Solving a partial differential equation by writing the unknown as a product of single-variable factors.

stability condition: The step-size restriction that keeps a numerical recurrence from amplifying errors.