

Quantum occupations and limits of equipartition

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Quantum occupations and limits of equipartition

A fermionic energy level can contain several particles when it contains several distinct states; the exclusion rule applies per complete state.

Prerequisites: 20, 35, 38.

- Compute mean occupation $\bar{n}$ per complete quantum state for fermions and bosons
- Count allowed identical-particle occupations and identify the classical dilute limit
- Compare classical quadratic-mode heat capacity with a quantum oscillator response

Assign occupation to complete states

For noninteracting particles in thermal and particle exchange equilibrium, use chemical potential μ and $x = (\epsilon - \mu)/(k_B T)$. Mean occupation of one complete state is $\bar{n}_F = 1/(\hat{e}^x + 1)$ for fermions and $\bar{n}_B = 1/(\hat{e}^x - 1)$ for bosons. Fermionic occupation of that state is zero or one; its mean lies between them. Bosonic occupation can exceed one. A level of degeneracy g has total mean g times the single-state occupation when its states share the same energy. Count spin as part of a complete state. The Bose denominator requires $\epsilon > \mu$ for the ordinary finite expression, with the ground-state limit treated separately. For equilibrium photons $\mu = 0$ because photon number is not conserved; do not set $\mu = 0$ for every material particle gas.

Count indistinguishable configurations

Count occupation patterns rather than labelling identical particles. Two identical fermions distributed among four distinct complete states have $\text{choose}(4, 2) = 6$ allowed patterns. Two identical bosons among those states have $\text{choose}(4 + 2 - 1, 2) = 10$ patterns because both may share one state. Two labelled distinguishable particles would have $4^2 = 16$ assignments. These are different counting models, not three interchangeable answers to the same specification. If a question supplies spin degeneracy, first decide whether its stated number counts complete states or just orbital levels. A Pauli prohibition on two identical complete states does not prohibit opposite-spin fermions in one spatial orbital.

Check the dilute approximation

When x is large and positive, occupation is small and both denominators are dominated by $\hat{e}^x$: $\bar{n}_F \approx \bar{n}_B \approx \hat{e}^{-x}$, the Maxwell–Boltzmann dilute limit. At $x = \ln 4$ the means are $1/5$ for fermions, $1/3$ for bosons and $1/4$ in the classical approximation; the difference is still significant. A large total particle number alone does not justify classical statistics: density, temperature and accessible states control occupation. For fermions at low T , states below μ become nearly occupied and those above nearly empty. At $\epsilon = \mu$ a fermionic state has mean one half; inserting that value into the ordinary Bose formula would instead produce a divergent denominator and requires different limiting treatment.

Test quantum versus classical modes

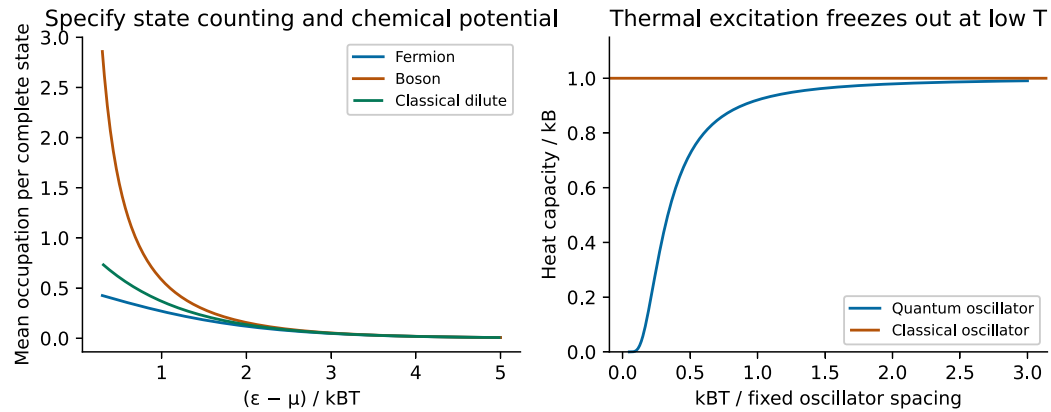
Classical equipartition assigns $k_B T/2$ of mean energy to each independent quadratic term in an equilibrated Hamiltonian. A monatomic ideal gas has three translational terms, giving $U = 3Nk_B T/2$ and $C_V = 3Nk_B/2$. One classical one-dimensional harmonic oscillator has kinetic and potential terms, giving mean $k_B T$ and $C = k_B$. For a quantum oscillator with fixed spacing $\epsilon = \hbar\omega$, the thermal energy above its temperature-independent zero point is $\epsilon/(\hat{e}^x - 1)$, now $x = \epsilon/(k_B T)$, and $C/k_B = x^2 \hat{e}^x / (\hat{e}^x - 1)^2$. At high T it approaches the classical value; at low T excitation freezes out and $C \rightarrow 0$. A zero-point energy $\epsilon/2$ shifts U but not C . Molecular rotational/vibrational contributions likewise need their energy scales checked before assigning classical quadratic terms.

Worked method

Occupation number $\bar{n}$ refers to one complete single-particle state. For $x = (\epsilon - \mu)/(k_B T) = \ln 4$,

$$\bar{n}_F = (\hat{e}^x + 1)^{-1} = 1/5, \quad \bar{n}_B = (\hat{e}^x - 1)^{-1} = 1/3.$$

The classical dilute approximation gives $\hat{e}^{-x} = 1/4$; it is not exact here. A g -fold level has mean total occupation $g\bar{n}$, so a fermionic level can contain more than one particle while each complete state still obeys exclusion.



Quantum occupations and limits of equipartition: original GRE teaching diagram.

Check conditions and vocabulary

Exclude two fermions from one complete state, not from an entire degenerate energy level. State which x is used; chemical-potential occupations and fixed oscillator excitation formulas are different models.

mean occupation: Ensemble average number of particles in one complete quantum state or a specified group of states.

equipartition: Classical equilibrium rule assigning $kBT/2$ to each independent quadratic Hamiltonian term.