

Speed distributions and statistical ensembles

GRE · GRE Subject Test · GRE Physics

A gas can have zero mean velocity while its particles have substantial average speed. Opposite directions cancel in a vector mean; speed has no sign.

Prerequisites: 4, 35.

- Normalise continuous speed densities and interpret their units
- Distinguish vector means, speed means and most probable speeds
- Use Boltzmann weights with degeneracy and a partition function

Choose the system and model

For a continuous speed probability density $f(v)$, $f(v)dv$ approximates probability in a narrow interval and $\int_{0\text{ to } \infty} f(v)dv=1$. The density has inverse-speed units and is not itself a probability. If $P(v)dv$ counts particles in that interval, its area is the particle count N and $f=P/N$. A trapezoid rising over width v_0 , remaining flat over width $2v_0$, then falling over width v_0 has area $3av_0$ for height a . Normalise by area before computing a mean or interpreting a plotted height.

Use the governing relation

In an isotropic equilibrium gas without bulk flow, each velocity-component distribution is symmetric, so $\langle v_{\text{vector}} \rangle = 0$. Speed $v = |v_{\text{vector}}|$ is nonnegative and has positive mean. For a classical ideal gas, the Maxwell speed density is proportional to $v^2 \exp[-mv^2/(2k_B T)]$. Its mode is $\sqrt{2k_B T/m}$, its mean is $\sqrt{8k_B T/(\pi m)}$ and its rms speed is $\sqrt{3k_B T/m}$. These are three different quantities. The v^2 factor makes speed density zero at $v=0$, even though the velocity-vector density is largest at the zero vector.

Apply the conditions

For a continuous distribution, the probability of exactly one specified speed is zero; nonzero probabilities refer to intervals. This statement does not imply there are no particles with arbitrarily small speeds, or that a finite-resolution detector cannot record a zero bin. Temperature changes the scale of the Maxwell distribution: characteristic speeds are proportional to $\sqrt{T/m}$. Doubling temperature does not double the speed, and heavier particles are slower on average at the same temperature.

Check the conclusion

In a canonical ensemble at temperature T , a state of energy E_i has weight $\exp(-E_i/(k_B T))$. Divide by partition function $Z = \sum \exp(-E_i/(k_B T))$ to obtain probabilities. If an energy level has degeneracy g_i , its level probability is $g_i \exp(-E_i/(k_B T))/Z$. Equal energy per state does not imply equal probability per level when degeneracies differ. For a ground level of degeneracy 1 and an excited level of degeneracy 2 at $\Delta = k_B T \ln 2$, the excited total weight is $2e^{-(\ln 2)} = 1$: the two levels each have probability $1/2$. Classical Maxwell–Boltzmann assumptions differ from Bose–Einstein and Fermi–Dirac quantum occupations.

Worked method

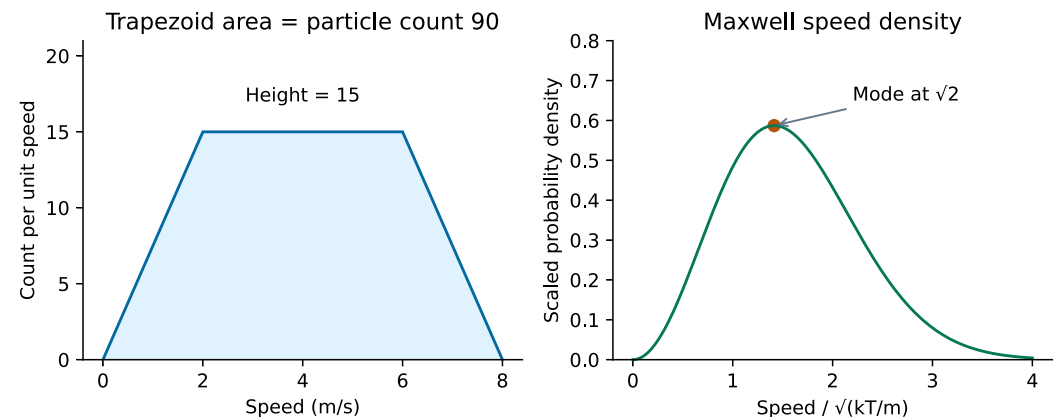
A canonical ensemble has ground degeneracy 1 and excited degeneracy g at gap Δ .

$$Z = 1 + g e^{-\Delta/(k_B T)}, \quad P_e = \frac{g e^{-\Delta/(k_B T)}}{Z}.$$

With $g = 4$ and $\Delta = k_B T \ln 2$,

$$P_e = \frac{4e^{-\ln 2}}{1 + 4e^{-\ln 2}} = \frac{2}{3}.$$

This is probability for the entire excited level. Each of its four states has probability $1/6$.



Speed distributions and statistical ensembles: original GRE teaching diagram.

Check conditions and vocabulary

A density height is not a probability. Mean velocity, mean speed and mode differ; include degeneracy before normalising level probabilities.

probability density: Probability per unit of a continuous variable, whose integral gives interval probability.

partition function: Sum of statistical weights used to normalise equilibrium state probabilities.