

26.1 Constraints, variational equations and cyclic coordinates

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS holonomic constraint • cyclic coordinate

Objective

Derive Euler–Lagrange motion after imposing a holonomic constraint _____; Use a cyclic coordinate _____ to identify conserved canonical momentum; Linearise a stable equilibrium and check the resulting frequency. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

Use a fixed-length pendulum constraint $x=l \sin\theta$, $y=-l \cos\theta$. Both velocity components give $T=ml^2\dot{\theta}^2/2$; $V=mgl(1-\cos\theta)$. A holonomic constraint relates coordinates, not merely arbitrary velocities.

With ideal constraints, $L=T-V$ and $d(\partial L/\partial \dot{q})/dt - \partial L/\partial q = 0$. The pendulum equation is $ml^2\ddot{\theta} + mgl \sin\theta = 0$. Differentiate all time-dependent factors.

Central motion has $L=m(\dot{r}^2 + r^2\dot{\theta}^2)/2 - V(r)$. θ is cyclic, so $mr^2\dot{\theta}$ is constant; $m\ddot{r} = mr\dot{\theta}^2 - V'(r)$. Conserved angular momentum does not imply constant angular speed.

Near a stable equilibrium, $\omega^2 = V''/M$ for the local inertia M . The pendulum approximation $\sin\theta \approx \theta$ requires radians and small amplitude; $\omega = \sqrt{g/l}$. Negative curvature indicates instability.

■ Worked method

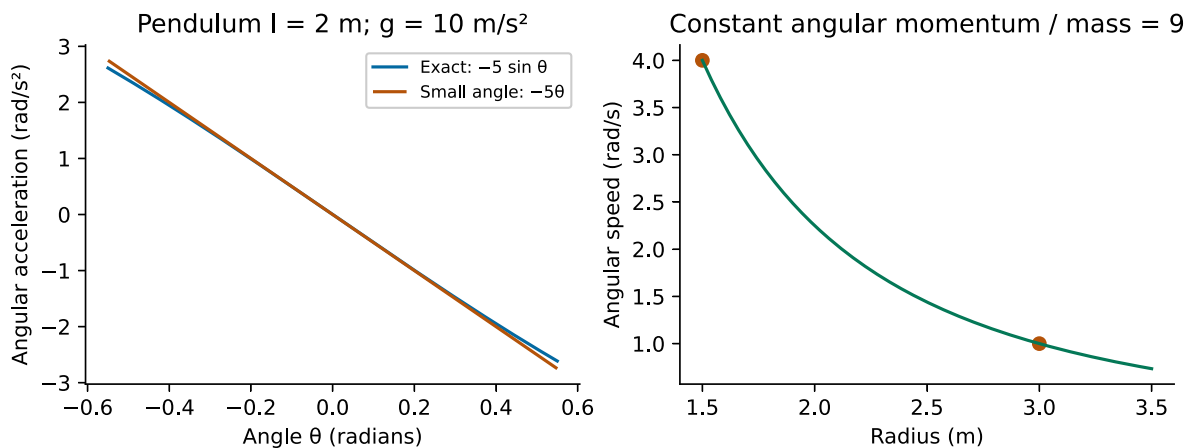
For a fixed pendulum length l , use θ from downward vertical: $x = l \sin \theta$, $y = -l \cos \theta$. Both Cartesian velocities give $T = ml^2\dot{\theta}^2/2$ and $V = mgl(1 - \cos \theta)$. The Euler-Lagrange equation _____ is

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl \sin \theta = 0.$$

Divide by ml^2 before linearising:

$$\ddot{\theta} + (g/l) \sin \theta = 0, \quad \omega = \sqrt{g/l} \quad (|\theta| \ll 1).$$

The approximation uses radians. Endpoint variations vanish in the stationary-action derivation.



2 Practice

2.1 Choose one answer. For a pendulum angle θ measured from downward vertical, which kinetic energy uses both Cartesian velocity components? [1]

- **A** $ml^2 \sin^2 \theta \dot{\theta}^2 / 2$
- **B** $ml^2 \cos^2 \theta \dot{\theta}^2 / 2$
- **C** $ml^2 \dot{\theta}^2 / 2$
- **D** $mgl\theta^2 / 2$
- **E** $ml\dot{\theta} / 2$

2.2 Choose one answer. In $L = m(r\dot{\theta}^2 + r^2\dot{\theta}^2) / 2 - V(r)$, which statement follows from the cyclic angle? [1]

- **A** θ is constant
- **B** $\dot{\theta}$ is constant
- **C** r is constant
- **D** $mr^2\dot{\theta}$ is constant
- **E** Mechanical kinetic energy is constant

2.3 For $V = 9q^2 / 2$ and local inertia $M = 1$, small-oscillation angular frequency is _____ rad/s. Show your relation and units. [2]

Solutions

2.1 C Adding $\dot{x}^2$ and $\dot{y}^2$ gives $l^2\dot{\theta}^2$.

Compare the rejected options: $ml^2 \sin^2\theta \dot{\theta}^2/2$; $ml^2 \cos^2\theta \dot{\theta}^2/2$; $mg l\theta^2/2$; $ml\dot{\theta}/2$. Use the stated model and conditions.

2.2 D $\partial L/\partial\theta=0$ gives $d(mr^2\dot{\theta})/dt=0$; radius and angular speed may change.

Compare the rejected options: θ is constant; $\dot{\theta}$ is constant; r is constant; Mechanical kinetic energy is constant. Use the stated model and conditions.

2.3 3. $V''=9$ and $\sqrt{9/1}=3$.

2.4 False. $\sin\theta\approx\theta$ requires radians; a degree value needs conversion.

3.1 Prescribed rotation gives $L = m(\dot{r}^2 + \Omega^2 r^2)/2$. Euler-Lagrange gives $\ddot{r} = \Omega^2 r$. The time-independent reduced L has conserved $\dot{r} \partial L/\partial \dot{r} - L = m(\dot{r}^2 - \Omega^2 r^2)/2$. Mechanical energy is $m(\dot{r}^2 + \Omega^2 r^2)/2$ and is not generally conserved: the rotation drive supplies work through the constraint. Award L , equation, conserved quantity, mechanical-energy distinction and drive. The bead's angular momentum is not conserved.