

22.1 Relativistic lifetime, energy and Doppler shift

Name: _____ Class: _____ Date: _____ **Total: 10 marks**

KEY WORDS proper time • rest energy

Objective

Relate proper lifetime to laboratory time and distance; Use invariant energy–momentum and relativistic kinetic energy; Infer longitudinal recession speed from wavelength ratio. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

For relative speed v , define $\beta=v/c$ and $\gamma=1/\sqrt{1-\beta^2}$. Proper time _____ is measured along the particle’s worldline by a clock at rest with it. If its proper mean lifetime is τ_0 , its laboratory mean lifetime is $\gamma\tau_0$ and mean travel distance is $v\gamma\tau_0$ at constant speed. The decay is statistical: mean lifetime is not a guaranteed decay time for each particle. At $\beta=0.8$, $\gamma=5/3$. Compute spacetime events consistently in one frame; proper time and coordinate time are different quantities.

Total energy _____ is $E=\gamma mc^2$, momentum $p=\gamma mv$ and invariant $E^2-p^2c^2=m^2c^4$. When E and pc are expressed in the same energy units, find $mc^2=\sqrt{E^2-(pc)^2}$, not $E-pc$. The nonnegative square root is required for positive rest mass. For $E=13\text{ GeV}$ and $pc=12\text{ GeV}$, rest energy _____ is 5 GeV and mass is $5\text{ GeV}/c^2$. A massless particle has $E=pc$ but need not have zero energy or momentum. These relations concern isolated-particle four-momentum, not classical mv at relativistic speed.

Work accelerating a particle from rest equals kinetic energy $K=E-mc^2=(\gamma-1)mc^2$. At $\beta=0.6$, $\gamma=1.25$ and $K=0.25mc^2$, while the classical $\frac{1}{2}mv^2$ gives $0.18mc^2$. Classical kinetic energy is the low-speed expansion and becomes inaccurate near c . Finite acceleration work increases γ rather than allowing a massive particle to reach or exceed c . Keep total energy, rest energy and kinetic energy distinct when interpreting answer units.

For purely longitudinal relative recession in special relativity, wavelength ratio $r=\lambda_{\text{observed}}/\lambda_{\text{emitted}}$. Rearranging gives $\beta=(r^2-1)/(r^2+1)$. A ratio $r=2$ gives $\beta=3/5$, not c times $r-1$ from a low-speed approximation. Blueshift uses $r<1$ and a negative recession parameter under this convention. The formula assumes the shift is entirely kinematic; cosmological expansion or gravitational redshift requires a different model. State the question’s stipulated model before interpreting a spectral ratio.

■ Worked method

Total energy and momentum determine invariant rest energy.

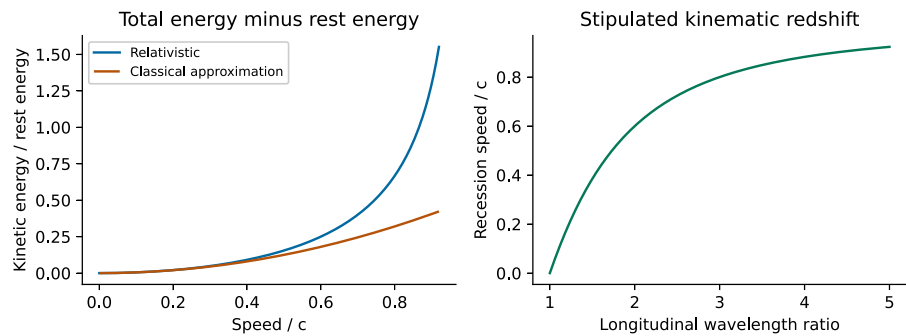
$$E^2-(pc)^2=(mc^2)^2, \quad mc^2=\sqrt{E^2-(pc)^2}.$$

For $E=17\text{ GeV}$ and $pc=15\text{ GeV}$,

$$mc^2=\sqrt{(17\text{ GeV})^2-(15\text{ GeV})^2}=8\text{ GeV}.$$

$$K=E-mc^2=17\text{ GeV}-8\text{ GeV}=9\text{ GeV}.$$

Rest mass is $8\text{ GeV}/c$ squared. Do not subtract momentum directly from energy.



2 Practice

2.1 Choose one answer. A particle at $0.8c$ has proper mean lifetime $6\text{ }\mu\text{s}$. Its laboratory mean lifetime is: [1]

- **A** $3.6\text{ }\mu\text{s}$
- **B** $4.8\text{ }\mu\text{s}$
- **C** $6\text{ }\mu\text{s}$
- **D** $10\text{ }\mu\text{s}$
- **E** $30\text{ }\mu\text{s}$

2.2 Choose one answer. For $E=10\text{ GeV}$ and $pc=6\text{ GeV}$, rest energy is: [1]

- **A** 4 GeV
- **B** 6 GeV
- **C** 8 GeV
- **D** 10 GeV
- **E** 16 GeV

2.3 At speed $0.6c$, $K/(mc^2) = \underline{\hspace{2cm}}$ (decimal). Show your relation and units. [2]

2.4 Decide whether this claim is true. Justify it: A wavelength redshift ratio of 2 implies superluminal recession under the stipulated longitudinal special-relativity model. [2]

3 Exam-style questions

3.1 Equal-energy photons of energy E collide with directions separated by angle θ . Derive the invariant mass of the pair. Check parallel and head-on limits. [4]

Solutions

2.1 D $\gamma=5/3$, so $\gamma\tau_0=10\ \mu\text{s}$.

Compare the rejected options: $3.6\ \mu\text{s}$; $4.8\ \mu\text{s}$; $6\ \mu\text{s}$; $30\ \mu\text{s}$. Use the stated model and conditions.

2.2 C $\sqrt{100-36}=8\ \text{GeV}$.

Compare the rejected options: $4\ \text{GeV}$; $6\ \text{GeV}$; $10\ \text{GeV}$; $16\ \text{GeV}$. Use the stated model and conditions.

2.3 0.25. $\gamma=1.25$, so $\gamma-1=0.25$.

2.4 False. $\beta=(4-1)/(4+1)=0.6$.

3.1 Total energy is $2E$; total momentum magnitude is $2(E/c)|\cos(\theta/2)|$. $M^2c^4 = 4E^2 - 4E^2\cos^2(\theta/2) = 4E^2\sin^2(\theta/2)$. Thus $Mc^2 = 2E\sin(\theta/2)$ for $0 \leq \theta \leq \pi$. Parallel photons have zero pair invariant mass; head-on photons have $Mc^2 = 2E$. Award vector momentum, invariant, result and limits. Each photon remains massless.