

Exercise walk-through

38.1

Exchange symmetry, spin pairs and Pauli exclusion

GRE · GRE Subject Test · GRE Physics

38.1 Exchange symmetry, spin pairs and Pauli exclusion

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS spin singlet • Slater determinant

Objective

Construct normalised symmetric and antisymmetric two-orbital spatial states; Combine spatial and spin symmetry to produce allowed two-electron states; Count complete state occupations and interpret exchange-induced correlations. Original public practice; not an ETS paper.

1 Key knowledge and worked examples

For orthonormal distinct orbitals, $\Psi_{\pm} = [a(1)b(2) \pm b(1)a(2)]/\sqrt{2}$. Fermionic total states are antisymmetric under simultaneous exchange of spatial and spin labels. Identical-orbital normalisation needs a separate treatment.

The spin singlet is antisymmetric and pairs with symmetric electron space. Triplet spin is symmetric and pairs with antisymmetric space. Same-orbital double occupation permits the singlet, not a triplet.

Two orbitals $\times$ two spin states give four complete states and choose(4,2)=6 two-electron occupations. Three orbitals give choose(6,2)=15. Exclusion is per complete state, not per spatial orbital.

Antisymmetric spatial density vanishes at $x_1=x_2$. Exchange interference changes correlations without introducing a classical force. Joint position density has units inverse length squared and must be integrated for an interval probability.

■ Worked method

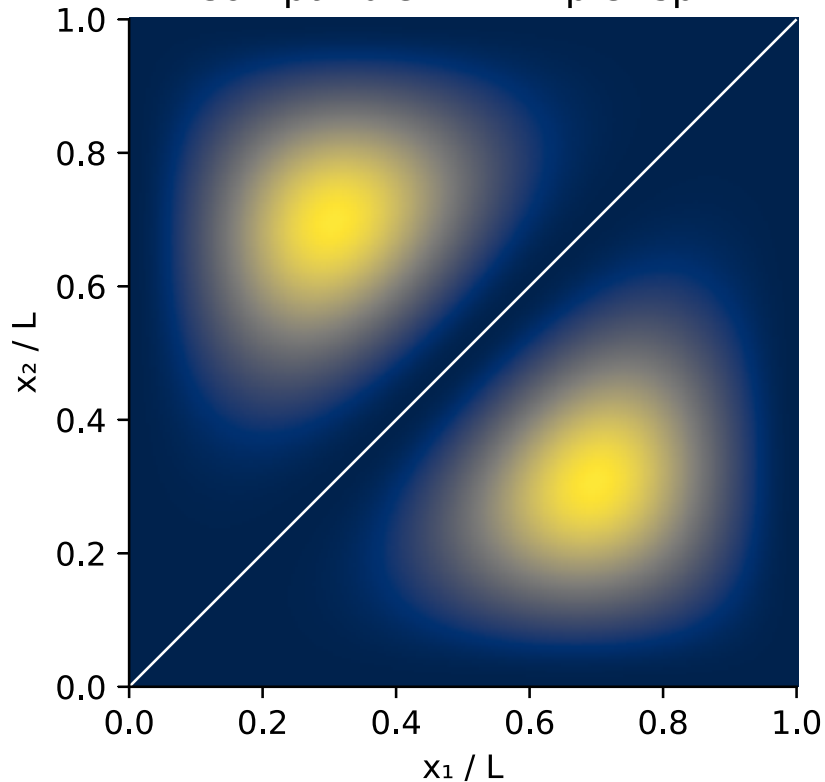
For orthonormal different orbitals a and b, symmetric and antisymmetric spatial states

are

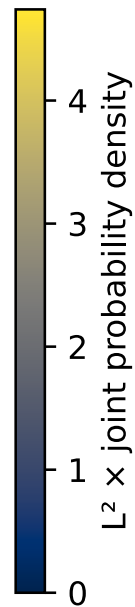
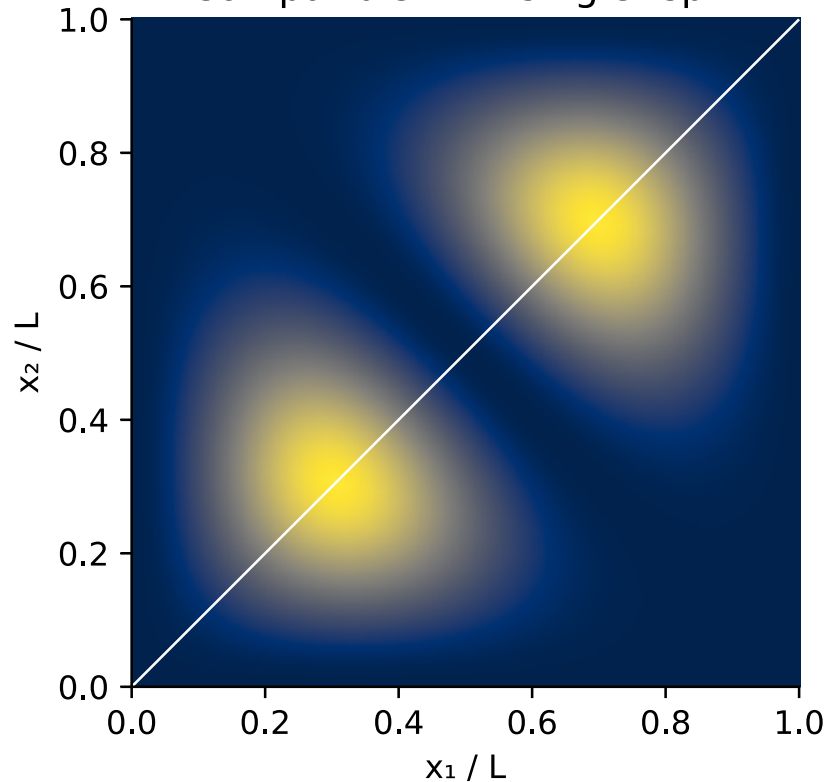
$$\Psi_{\pm} = [a(1)b(2) \pm b(1)a(2)]/\sqrt{2}.$$

A two-electron total state must be antisymmetric under simultaneous exchange of position and spin. Thus symmetric space pairs with the spin singlet; antisymmetric space pairs with triplet spin. If both electrons occupy a, the spatial product is symmetric and only singlet spin is allowed. The different-orbital normalisation formula cannot simply be reused for a = b.

Antisymmetric space
Compatible with triplet spin



Symmetric space
Compatible with singlet spin



Exercise 2.1

2.1 Choose one answer. Two electrons share the same spatial orbital. Which spin state is allowed for their antisymmetric total state? [1]

- **A** $\uparrow\uparrow$
- **B** $\downarrow\downarrow$
- **C** Symmetric $\uparrow\downarrow + \downarrow\uparrow$ triplet
- **D** Antisymmetric $\uparrow\downarrow - \downarrow\uparrow$ singlet
- **E** Any spin state

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- **D** Antisymmetric $\uparrow\downarrow - \downarrow\uparrow$ singlet
- **E** Any spin state

2.1 D The same-orbital spatial product is symmetric; the spin factor must be antisymmetric.

Compare the rejected options: $\uparrow\uparrow$; $\downarrow\downarrow$; Symmetric $\uparrow\downarrow + \downarrow\uparrow$ triplet; Any spin state. Use the stated model and conditions.

Exercises 2.2·2.3

2.2 Choose one answer. Two spinless identical bosons can use two spatial orbitals without an energy restriction. Occupation patterns number: [1]

- A 1
- B 2
- C 3
- D 4
- E 6

2.3 Two electrons access three spatial orbitals, each with two spin states. Complete-state occupation patterns number _____. Show your relation and units. [2]

Exercises 2.2-2.3

2.2 Choose one answer. Two spinless identical bosons can use two spatial orbitals without an energy restriction. Occupation patterns number: [1]

- A 1
- B 2
- C 3
- D 4
- E 6

2.2 C Allowed patterns are (2,0), (1,1) and (0,2); particle labels do not create separate versions of (1,1).

Compare the rejected options: 1; 2; 4; 6. Use the stated model and conditions.

2.3 Two electrons access three spatial orbitals, each with two spin states. Complete-state occupation patterns number _____. Show your relation and units. [2]

2.3 15. Six complete states give $\text{choose}(6,2)=15$.

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: An antisymmetric two-orbital spatial state must have zero amplitude when the two spatial arguments coincide. [2]

3.1 Two electrons access three orthonormal spatial orbitals, with no energy restriction. Count singlet states and triplet states, including spin projections. Reconcile the sum with choosing two complete states from six. [5]

Exercises 2.4·3.1

2.4 Decide whether this claim is true. Justify it: An antisymmetric two-orbital spatial state must have zero amplitude when the two spatial arguments coincide. [2]

2.4 True. At $x_1=x_2$ the two products are equal and their difference vanishes.

3.1 Two electrons access three orthonormal spatial orbitals, with no energy restriction. Count singlet states and triplet states, including spin projections. Reconcile the sum with choosing two complete states from six. [5]

3.1 Symmetric spatial states: three double occupations plus three different-orbital symmetric pairs, giving six singlets. Antisymmetric space has three orbital pairs; each permits three triplet spin projections, giving nine triplet states. Total is 15, equal to $\binom{6}{2}$. Award double occupations, singlet total, antisymmetric pairs, triplet total and reconciliation. Counting only the three triplet multiplets would omit their spin projections.