

T.12 · Metric completeness and closure from a basis

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Metric axioms, Cauchy sequences and topology from a basis.

3. Scope and prerequisites

Scope and prerequisites

Check a pullback metric using the properties of its defining map Decide completeness through the image of an isometry Determine closure using every basic neighbourhood instead of Euclidean intuition

4. Scope and prerequisites

Scope and prerequisites

complete metric space : A metric space in which every Cauchy sequence converges to a point of that space.

5. Scope and prerequisites

Scope and prerequisites

neighbourhood basis : Basic neighbourhoods sufficient to test local topological properties.

6. Choose and justify a method

Choose and justify a method

For an injective map $h:X\rightarrow\mathbb{R}$, $d(x,y)=|h(x)-h(y)|$ is a metric: symmetry and the triangle inequality come from $\mathbb{R}$, and injectivity ensures $d(x,y)=0$ only when $x=y$. The map h is an isometry onto its image $h(X)$. If h is not injective, this construction may be only a pseudometric. Bounded distances do not prove a space complete, and a familiar set of points can have very different Cauchy behaviour under different metrics.

7. Choose and justify a method

Choose and justify a method

The pullback metric is complete exactly when the image $h(X)$ is complete in the ordinary real distance. A closed subset of $\mathbb{R}$ is complete. For $h(x) = \arctan x$ on $\mathbb{R}$, the image is $(-\pi/2, \pi/2)$, which omits its endpoints. The sequence n is Cauchy in the pullback metric because $\arctan n \rightarrow \pi/2$, but no real x has $\arctan x = \pi/2$, so the metric is incomplete. For $h(x) = x^3$, the image is all $\mathbb{R}$ and the pullback metric is complete. Topological equivalence alone does not preserve completeness.

8. Choose and justify a method

Choose and justify a method

In a topology with a basis, x lies in the closure of A when every basic open neighbourhood of x meets A . This is a membership test, not automatically a Euclidean endpoint operation. The basis must cover the space, and for a point in two basis sets there must be a smaller basis set containing it within their intersection. A point can lie in the closure of a singleton even when it is not the singleton's own point; that possibility depends on the topology.

9. Choose and justify a method

Choose and justify a method

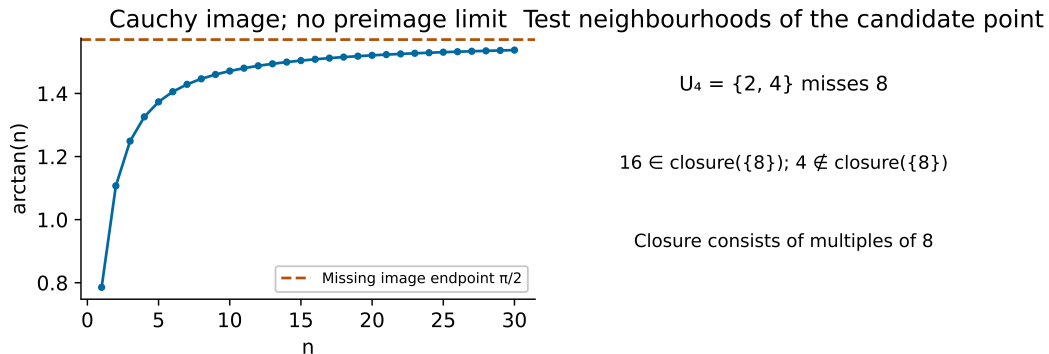
On $X = \{2, 3, 4, \dots\}$, take basic sets $U_k = \{n \text{ in } X: n \text{ divides } k\}$, for $k \geq 2$. They cover X since $x \in U_x$; intersections are $U_{\gcd(k, l)}$ when the gcd is at least 2, or empty. A point x lies in closure of $\{n\}$ exactly when every k divisible by x is also divisible by n : equivalently n divides x . More directly, the smallest basic neighbourhood U_x contains n precisely when n divides x , and every other neighbourhood containing x includes these divisors. Thus closure of $\{8\}$ consists of positive multiples of 8, not the divisors of 8.

10. Worked reasoning

Worked reasoning

Under $d(x,y)=|\arctan x-\arctan y|$, the integer sequence goes toward a missing image endpoint rather than a point of $\mathbb{R}$, so it is Cauchy without convergence in that space. In the divisor basis, 16 lies in closure of $\{8\}$, since any basic set containing 16 also contains 8. But 4 does not: $U_4=\{2,4\}$ is a neighbourhood of 4 missing 8. This explicitly separates multiples from divisors.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

A Cauchy limit must belong to the same space. Completeness is a metric property, not just a topological one. For closure, test every neighbourhood of the candidate point, rather than every neighbourhood of the singleton value.

13. Try independently

Try independently

On $\mathbb{R}$ compare $d_1(x, y) = |x^3 - y^3|$ and $d_2(x, y) = |\arctan x - \arctan y|$. Prove they are metrics and decide completeness.

14. Reasoning and check

Reasoning and check

Both defining maps are injective, giving positive separation; absolute distance gives symmetry and the triangle inequality. Each map is an isometry onto its real image. The cube map has image all of $\mathbb{R}$, a complete space, so d_1 is complete. The arctangent image is the open interval $(-\pi/2, \pi/2)$, which is incomplete. The sequence n is d_2 -Cauchy because $\arctan n$ tends to $\pi/2$, but no real point has that image, so it has no d_2 -limit. Bounded metric values do not imply completeness.

15. Try independently

Try independently

On $X = \{2, 3, 4, \dots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections.

16. Reasoning and check

Reasoning and check

Every x belongs to U_x . An intersection is $U_{\gcd(k,l)}$ if the gcd is at least two, and is empty for gcd one. Thus a point in an intersection has a basic neighbourhood within it. The smallest basic neighbourhood of x is U_x , contained in every U_k that contains x . It meets $\{6\}$ exactly when $6 \mid x$. Therefore the closure is $\{6, 12, 18, \dots\}$, the multiples of six. The point six is not interior to its singleton because U_6 also contains two and three. No other point is in that singleton, so its interior is empty.

17. Reasoning and check

Reasoning and check

This closure calculation uses all basic neighbourhoods, not Euclidean intuition.

Check yourself

Reconstruct the method without looking back

On $X = \{2, 3, 4, \dots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections.