

T.10 · Vector geometry, projections and oriented area

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Coordinate vectors, dot/cross products and norms.

3. Scope and prerequisites

Scope and prerequisites

Use dot products to classify angles and compute projections Calculate triangle area and orientation using a cross product Construct or rule out planar dot-product sign configurations

4. Scope and prerequisites

Scope and prerequisites

orthogonal : Having a zero dot product in a real inner-product space.

5. Scope and prerequisites

Scope and prerequisites

cross product : An oriented perpendicular vector in three dimensions whose magnitude is spanned parallelogram area.

6. Choose and justify a method

Choose and justify a method

For nonzero real vectors u, v , $u \cdot v = |u||v| \cos \theta$. A positive, zero or negative dot product corresponds to an acute, right or obtuse smaller angle. The scalar projection of v along u is $(v \cdot u)/|u|$; its vector projection is $((v \cdot u)/(u \cdot u))u$. The difference from this projection is orthogonal to u . Do not confuse a projected vector with its signed scalar component. The zero vector is orthogonal to every vector but has no defined angle direction.

7. Choose and justify a method

Choose and justify a method

For three-dimensional vectors, $\mathbf{u} \times \mathbf{v}$ is perpendicular to both with length $|\mathbf{u}||\mathbf{v}| \sin \theta$. Coordinate calculation uses $(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$. This length is the parallelogram area, so a triangle from two edge vectors has half that area. Reversing their order reverses the cross product but preserves area. Build both edge vectors from the same vertex; crossing two unrelated position vectors generally measures the wrong triangle.

8. Choose and justify a method

Choose and justify a method

The plane through a point p with nonzero normal n has equation $n \cdot (x-p)=0$. Distance from q to the plane is $|n \cdot (q-p)|/|n|$; the denominator normalises the scale of the equation. A scalar triple product $u \cdot (v \times w)$ gives signed parallelepiped volume; its absolute value is geometric volume. Zero triple product means dependence of the three edge vectors, not necessarily that each pair is perpendicular or parallel.

9. Choose and justify a method

Choose and justify a method

For four planar vectors, there are six unordered dot products. The configuration $e_1, -e_1, e_2, -e_2$ has two negative products and four zeros; e_1, e_1, e_2, e_2 has two positive products and four zeros. Four nonzero vectors cannot have every pairwise dot product negative. Order their directions around the circle: each consecutive angular gap would have to exceed 90° , forcing the sum of four gaps above 360° . A zero vector cannot rescue a strict-negative requirement, because its dot products vanish.

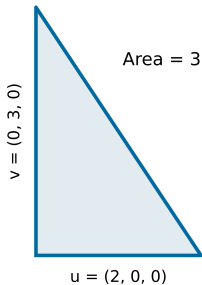
10. Worked reasoning

Worked reasoning

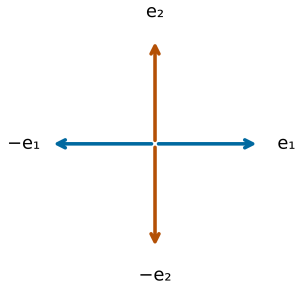
With $p=(0,0,0)$, $q=(2,0,0)$, $r=(0,3,0)$, the edges are $u=(2,0,0)$, $v=(0,3,0)$. Their cross product is $(0,0,6)$, giving triangle area 3. For $w=(3,4)$ and $u=(1,0)$, the vector projection is $(3,0)$ and the orthogonal remainder is $(0,4)$. The plane $2x-y+2z=6$ has normal length 3, so its distance from the origin is $6/3=2$.

Figure 11: Worked reasoning

Triangle area = $|u \times v| / 2 = 3$



Two negative products; four zeros



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

A cross-product magnitude is parallelogram area, so halve it for a triangle.

Dot-product zero is an algebraic orthogonality statement even for a zero vector. Plane distance must divide by normal length.

13. Try independently

Try independently

Project $v = (2, 3, 6)$ onto $u = (1, 0, 2)$. Find the area of the triangle with vertices O , u and v . Check the projection residual is orthogonal to u .

14. Reasoning and check

Reasoning and check

$v \cdot u = 14$ and $u \cdot u = 5$, so the vector projection is $(14/5)u = (14/5, 0, 28/5)$. The residual is $(-4/5, 3, 2/5)$, whose dot product with u is zero. The cross product is $u \times v = (-6, -2, 3)$ with norm seven. Triangle area is half the parallelogram area, or $7/2$. Both edge vectors start at O ; crossing unrelated position vectors would not generally compute the requested triangle.

15. Try independently

Try independently

Can four nonzero vectors in $\mathbb{R}^2$ have all six pairwise dot products strictly negative?
Give a proof. How does allowing zero products change the possibilities?

16. Reasoning and check

Reasoning and check

Order their directions around the circle. Every consecutive gap must exceed $\pi/2$: if a gap were at most $\pi/2$, that pair would have nonnegative dot product. Four such gaps would sum to more than 2π , impossible. Therefore not all six products can be strictly negative. With nonpositive products, $e_1, e_2, -e_1, -e_2$ works: consecutive pairs are perpendicular and opposite pairs have negative products. A zero vector cannot satisfy strict negativity.

Check yourself

Reconstruct the method without looking back

Can four nonzero vectors in $\mathbb{R}^2$ have all six pairwise dot products strictly negative?
Give a proof. How does allowing zero products change the possibilities?