

T.9 · Similarity, scaling and conic distance loci

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Similar triangles, distance formula and quadratic equations.

3. Scope and prerequisites

Scope and prerequisites

Match triangle vertices by equal angles before forming a side ratio Apply length, area and volume scale factors to geometric changes Identify conic distance conditions and retain a signed hyperbola branch

4. Scope and prerequisites

Scope and prerequisites

similarity : Equality of corresponding angles and a common ratio of corresponding lengths.

5. Scope and prerequisites

Scope and prerequisites

focus : A fixed point used in a conic distance definition.

6. Choose and justify a method

Choose and justify a method

Triangles are similar when their corresponding angles agree; their corresponding side ratios are then equal. Write the vertex correspondence explicitly. If triangle APQ has angle A equal to angle A of ABC and angle P equal to angle C, its ordered correspondence is $A \leftrightarrow A$, $P \leftrightarrow C$, $Q \leftrightarrow B$. Thus $AP/AC = AQ/AB = PQ/CB$. The side AP lies on AB but corresponds to AC; matching by where a side lies or by the letter P can give the wrong ratio.

7. Choose and justify a method

Choose and justify a method

With a common length scale factor $k > 0$, lengths multiply by k , areas by k^2 and volumes by k^3 . A 30% length increase means $k = 1.30$, hence an area increase of $1.30^2 - 1 = 0.69$, or 69%; it is not 60%. For similar solids with a volume ratio of 27, the length ratio is 3 and the surface-area ratio 9. Percentage changes and absolute differences are different quantities; write the new-to-old ratio before converting it to a percent.

8. Choose and justify a method

Choose and justify a method

A circle fixes distance to one point. An ellipse fixes the sum of distances to two foci. A hyperbola fixes the absolute difference of those distances; a parabola fixes equality of distance to a focus and to a directrix. These are definitions of point sets, not sketches to memorise. For foci $(-c,0)$ and $(c,0)$, an ellipse uses a sum $2a$ with $a > c$. A nondegenerate hyperbola uses an absolute difference $2a$ with $0 < a < c$; a signed difference distinguishes its two branches.

9. Choose and justify a method

Choose and justify a method

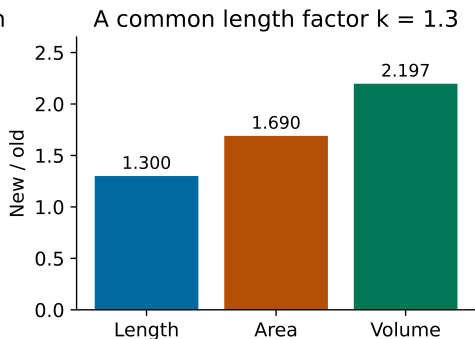
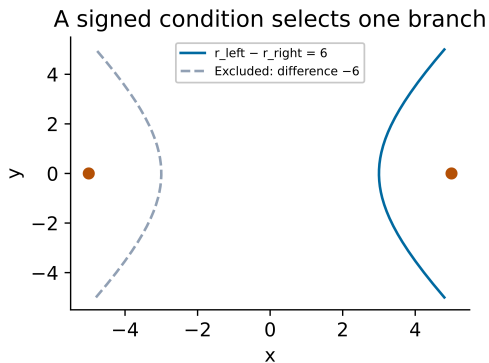
For the hyperbola with horizontal transverse axis, $c^2 = a^2 + b^2$ and the equation is $x^2/a^2 - y^2/b^2 = 1$. Let r_A be distance to $(-c, 0)$ and r_B distance to $(c, 0)$. The condition $r_A - r_B = 2a > 0$ selects the right branch $x \geq a$; the negative condition selects the left. Squaring can lose this sign, so check the original distance condition after obtaining the equation. Difference zero gives the perpendicular bisector rather than a hyperbola; a difference exceeding the focal separation is impossible by the triangle inequality.

10. Worked reasoning

Worked reasoning

Let $AB=10$, $AC=15$ and $BC=20$. A point P on AB has $AP=6$, and Q on AC makes angle APQ equal to angle ACB . Then $AP/AC=6/15=2/5$, so $PQ=8$ and $AQ=4$. Separately, foci at $(-5,0), (5,0)$ and distance-to-left minus distance-to-right equal to 6 give $a=3$, $b^2=25-9=16$: $x^2/9-y^2/16=1$ with $x \geq 3$. The point $(-3,0)$ satisfies the squared equation but fails the signed condition.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

Match angles before sides. Area percentages use the square of the length factor. A squared distance equation may describe both branches even when the original problem asks for only one.

13. Try independently

Try independently

Similar solids have surface-area ratio 9:25. Find their corresponding length and volume ratios. Explain why matching vertices matters before forming a triangle ratio.

14. Reasoning and check

Reasoning and check

Taking positive square roots gives length ratio 3:5. Cubing gives volume ratio 27:125. The area ratio is not itself a length scale. Corresponding vertices must match equal angles; otherwise ratios can compare unrelated sides. Similarity is an assumption here; area ratios alone do not prove that arbitrary solids are similar.

15. Try independently

Try independently

For foci $A = (-5, 0)$ and $B = (5, 0)$, find the locus $|PA| - |PB| = 6$. Derive its standard equation, choose its branch and check the vertex in the unsquared relation. What happens if the requested difference is 12?

16. Reasoning and check

Reasoning and check

Write $r_A = \sqrt{(x+5)^2 + y^2}$ and $r_B = \sqrt{(x-5)^2 + y^2}$. The original condition is $r_A - r_B = 6$. Since $r_A^2 - r_B^2 = 20x$, multiplying the distance difference by their sum gives $r_A + r_B = 10x/3$. Hence $r_B = 5x/3 - 3$, which must be nonnegative. Squaring this equality gives $(x-5)^2 + y^2 = (5x/3 - 3)^2$. Expanding and collecting gives $16x^2/9 - y^2 = 16$. Thus the half focal distance is $c=5$, half difference is $a=3$, and the hyperbola is $x^2/9 - y^2/16 = 1$, using $b^2 = c^2 - a^2 = 16$. The original positive difference selects the right branch $x \geq 3$. At

17. Reasoning and check

Reasoning and check

$(3, 0)$ the distances are 8 and 2, difference six; at $(-3, 0)$ the difference is negative six. Squaring alone retains this invalid branch. A difference of 12 exceeds $|AB| = 10$ and is impossible by the reverse triangle inequality.

Check yourself

Reconstruct the method without looking back

For foci $A = (-5, 0)$ and $B = (5, 0)$, find the locus $|PA| - |PB| = 6$. Derive its standard equation, choose its branch and check the vertex in the unsquared relation. What happens if the requested difference is 12?