

T.7 · Set images, equivalence relations and logical negation

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Set operations, quantifiers, functions and elementary proof.

3. Scope and prerequisites

Scope and prerequisites

Prove image identities and distinguish inclusion from equality Check reflexivity, symmetry and transitivity with explicit cases Negate implications and quantified statements without changing their scope

4. Scope and prerequisites

Scope and prerequisites

preimage : The set of inputs whose outputs lie in a specified target set.

5. Scope and prerequisites

Scope and prerequisites

equivalence relation : A relation that is reflexive, symmetric and transitive.

6. Choose and justify a method

Choose and justify a method

For $f:X \rightarrow Y$ and $A \subseteq X$, the image $f(A)$ consists of all outputs $f(a)$ with a in A . A point is in $f(A \cup B)$ exactly when it comes from A or B , so $f(A \cup B) = f(A) \cup f(B)$. If $A \subseteq B$, then $f(A) \subseteq f(B)$. For an intersection only $f(A \cap B) \subseteq f(A) \cap f(B)$ is automatic: the same output may come from different inputs. Use $f(x) = x^2$, $A = \{-1\}$, $B = \{1\}$; the left image is empty while the right intersection is $\{1\}$.

7. Choose and justify a method

Choose and justify a method

Preimages behave differently. For $C \subseteq Y$, $f^{-1}(C)$ here denotes the set of all inputs sent into C , even if f has no inverse function. Membership in two preimages means the very same input maps into both target sets. Consequently preimages preserve unions, intersections and complements relative to the stated domain/codomain. Do not transfer a theorem about preimages to images. If f is injective, image intersections do become equal, because equal outputs then force a shared input.

8. Choose and justify a method

Choose and justify a method

A relation R on X is reflexive when xRx for every x , symmetric when xRy implies yRx , and transitive when xRy and yRz imply xRz . All three make an equivalence relation; its classes partition X . On integers, xRy when x and y have the same remainder modulo four gives four classes. The relation $|x-y| \leq 1$ is reflexive and symmetric but not transitive: $0R1$ and $1R2$ while 0 is not related to 2 . Checking only a diagram or two properties is insufficient.

9. Choose and justify a method

Choose and justify a method

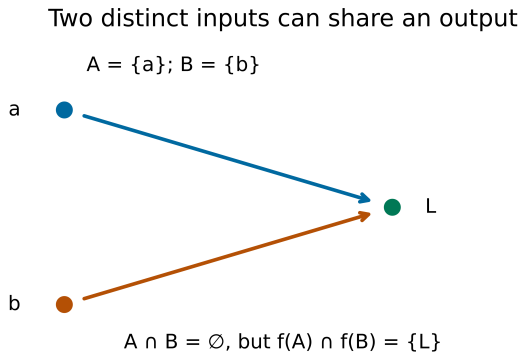
An implication $P \Rightarrow Q$ is false exactly when P is true and Q false. Thus the negation of $P \Rightarrow (Q \wedge R)$ is $P \wedge (\neg Q \vee \neg R)$, not $\neg P \Rightarrow (\neg Q \wedge \neg R)$. Negating “every x has property A ” gives “there exists x without A ”; negating “there exists x ” gives “every x does not”. A counterexample can disprove a universal claim, while examples cannot prove it. The quantifiers retain their order when individually negated: $\neg(\forall x \exists y S(x,y))$ is $\exists x \forall y \neg S(x,y)$.

10. Worked reasoning

Worked reasoning

Let f map both a and b to label L . With $A=\{a\}$, $B=\{b\}$, $f(A \cap B)=\emptyset$ but $f(A) \cap f(B)=\{L\}$. For xRy defined by $x=y$ or $x=-y$ on R , reflexivity and symmetry follow directly, and two sign changes still give $z=\pm x$, proving transitivity. The classes are $\{x, -x\}$, with $\{0\}$ a singleton. A false statement “every stored file has a hash” means at least one stored file has no hash; it does not mean all files lack hashes.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

$f^{-1}(C)$ can mean a preimage set without an inverse function. A reflexive, symmetric relation may still fail transitivity. Negate the whole statement before simplifying its parts.

13. Try independently

Try independently

Prove $f(A \cap B) \subseteq f(A) \cap f(B)$. Give a strict-inclusion example. Under what additional assumption on f does equality follow for all A and B ?

14. Reasoning and check

Reasoning and check

If $y = f(x)$ for $x \in A \cap B$, then x belongs to each set, so y belongs to each image. For strictness, map distinct a, b to the same c and take $A = \{a\}$, $B = \{b\}$. The left image is empty and the right intersection is $\{c\}$. If f is injective and $y = f(a) = f(b)$ with $a \in A, b \in B$, injectivity gives $a=b$ in the intersection, proving reverse inclusion. Conversely the singleton example shows that equality for all pairs forces injectivity.

15. Try independently

Try independently

On the integers define aRb when $a - b$ is divisible by 4. Prove it is an equivalence relation and describe every class. Negate: “For every integer n there exists an integer m greater than n with property P .”

16. Reasoning and check

Reasoning and check

Reflexivity uses $a - a = 0$; symmetry uses the negative of a multiple of four; transitivity uses the sum of two such multiples. Classes are the four residues modulo four; the class of a is $a + 4\mathbb{Z}$. The negation is: there exists an integer n such that every integer m greater than n fails P . Equivalently $\exists n \forall m (m > n \Rightarrow \neg P(m))$. It does not assert that P fails for every integer, nor that m is at most n for every m .

Check yourself

Reconstruct the method without looking back

On the integers define aRb when $a - b$ is divisible by 4. Prove it is an equivalence relation and describe every class. Negate: “For every integer n there exists an integer m greater than n with property P .”