

C.7 · Coordinate changes and vector integral theorems

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Multiple integrals, partial derivatives and oriented boundaries.

3. Scope and prerequisites

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Transform double and triple integrals with their Jacobians
Apply Green, Stokes and divergence theorems with correct orientation
Check domain singularities before asserting path independence

4. Scope and prerequisites

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divergence : The sum of a vector field's coordinate-wise partial derivatives measuring local outward flow.

5. Scope and prerequisites

Scope and prerequisites

conservative field : A vector field equal to a scalar potential gradient with path-independent line integrals.

6. Choose and justify a method

Choose and justify a method

Changing variables scales area or volume by the absolute Jacobian determinant. Polar coordinates use $r \, dr \, d\theta$; cylindrical coordinates use $r \, dr \, d\theta \, dz$; spherical coordinates with ϕ measured from the positive z -axis use $\rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta$. State angle conventions and transform both the integrand and the region. A missing Jacobian changes a uniform density integral into the wrong physical quantity.

7. Choose and justify a method

Choose and justify a method

Green's theorem equates positively oriented planar boundary circulation integral $\oint_C P dx + Q dy$ with the double integral of $Q_x - P_y$ on the region. The usual hypotheses require first derivatives continuous on an open set containing the region. A hole needs its own negatively oriented inner boundary, or another valid treatment of the missing domain. The theorem cannot integrate across a field singularity.

8. Choose and justify a method

Choose and justify a method

Stokes' theorem equates circulation on a surface boundary with the surface integral of $\text{curl } \mathbf{F}$ dot the oriented normal. The right-hand rule links boundary direction to the normal. The divergence theorem equates outward flux across a closed surface with the volume integral of $\text{div } \mathbf{F}$. Circulation, flux, curl and divergence are distinct; a closed surface is required for the usual divergence theorem.

9. Choose and justify a method

Choose and justify a method

A gradient field has path-independent line integrals, determined by endpoint potential differences. A continuously differentiable curl-free field on a simply connected open domain is conservative. Curl-free alone on a domain with a hole is insufficient. For $F=(-y/(x^2+y^2), x/(x^2+y^2))$, the origin is excluded; unit-circle circulation is 2π , although the curl is zero wherever the field is defined.

10. Worked reasoning

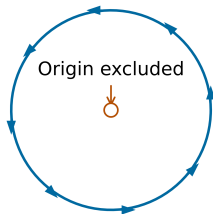
Worked reasoning

For $F=(x,y,z)$, divergence is 3. The outward flux through the sphere of radius 2 is therefore 3 times its volume, or $3 \cdot (4\pi \cdot 2^3/3)=32\pi$. This avoids a surface parameterisation. For the planar field $(-y,x)$, Green's theorem gives counterclockwise unit-circle circulation as integral of $1-(-1)=2$ over the disk, hence 2π . Reversing the orientation changes the circulation sign.

Figure 11: Worked reasoning

A hole can prevent a global potential

$$\mathbf{F} = (-y/r^2, x/r^2)$$



Unit-circle circulation = 2π , despite zero curl

Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

Zero curl is not enough when the domain has a hole. Outward flux and counterclockwise circulation use different theorems and orientation rules.

13. Try independently

Try independently

Evaluate $\iint_D (x^2 + y^2) dA$ on $1 \leq x^2 + y^2 \leq 4$. Find the outward flux of $F = (x, y, z)$ across the unit sphere.

14. Reasoning and check

Reasoning and check

The annulus has polar bounds $1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$. Including the Jacobian, $I = \int_0^{2\pi} \int_1^2 r^3 dr d\theta = 15\pi/2$. F is continuously differentiable on the whole ball and has divergence three. The divergence theorem gives outward flux $3(4\pi/3) = 4\pi$. Inward orientation would reverse its sign.

15. Try independently

Try independently

Let $G = (-y/(x^2 + y^2), x/(x^2 + y^2))$ on the punctured plane. Its scalar curl is zero there. Evaluate its circulation on the counterclockwise unit circle. Explain why Green's theorem does not make it zero.

16. Reasoning and check

Reasoning and check

Parameterise $r(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then $G(r(t)) = (-\sin t, \cos t) = r'(t)$, so $G \cdot r' = 1$ and the integral is 2π . The field is undefined at the origin inside the disk. Green's theorem requires the needed smoothness on a neighbourhood of the region, not just on its boundary. Zero curl on a punctured domain does not establish a global potential; a closed nonexact field can circulate around its hole.

Check yourself

Reconstruct the method without looking back

Let $G = (-y/(x^2 + y^2), x/(x^2 + y^2))$ on the punctured plane. Its scalar curl is zero there. Evaluate its circulation on the counterclockwise unit circle. Explain why Green's theorem does not make it zero.