

C.6 · Multivariable extrema and constrained optimisation

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Partial derivatives, quadratic forms, compactness and constraints.

3. Scope and prerequisites

Scope and prerequisites

Classify two-variable critical points with the Hessian Use Lagrange multipliers with a regular constraint Compare extrema and attainable values under regular constraints

4. Scope and prerequisites

Scope and prerequisites

Hessian : The matrix of second partial derivatives used to study local curvature.

5. Scope and prerequisites

Scope and prerequisites

Lagrange multiplier : A scalar relating objective and regular constraint gradients at a constrained extremum.

6. Choose and justify a method

Choose and justify a method

For a differentiable function on an open domain, an interior local extremum has zero gradient. This condition is necessary, not sufficient. For two variables at a critical point, let $D = f_{xx} f_{yy} - (f_{xy})^2$. If $D > 0$, $f_{xx} > 0$ gives a strict local minimum and $f_{xx} < 0$ gives a strict local maximum; $D < 0$ gives a saddle. These tests require suitable second-derivative regularity near the point.

7. Choose and justify a method

Choose and justify a method

When $D=0$ the second-derivative test is inconclusive, not evidence of a saddle. The functions x^4+y^4 and x^4-y^4 have the same zero Hessian at the origin, but one has a strict minimum and the other changes sign. Evaluate along contrasting directions or use a direct inequality. For a quadratic form, positive definiteness gives a more general Hessian interpretation.

8. Choose and justify a method

Choose and justify a method

On a smooth equality constraint $g(x,y)=c$ with nonzero gradient g , solve gradient $f=\lambda$ gradient g together with $g=c$. These equations generate candidates; they do not classify or guarantee a global extremum. If the constraint gradient vanishes, the regularity condition fails and the multiplier equations can miss a constrained extremum. Treat such points separately. For $x^2+y^2=a$ and $xy=b>0$, $(x-y)^2=a-2b$ shows necessity $a\geq 2b$. It is also sufficient: take $s=\sqrt{a+2b}$, $d=\sqrt{a-2b}$, $x=(s+d)/2$ and $y=(s-d)/2$. These real values have $xy=b$ and $x^2+y^2=a$.

9. Choose and justify a method

Choose and justify a method

A minimum condition alone needs this attainment check to establish solvability.

10. Choose and justify a method

Choose and justify a method

A continuous function on a compact feasible set attains global extrema. To find them, compare all interior candidates and all boundary pieces, including corners and endpoints. For a rectangle, optimise the restrictions on each edge. For a disk, the circular boundary may use a parameter or multiplier. Solving only the unconstrained gradient ignores possible boundary winners.

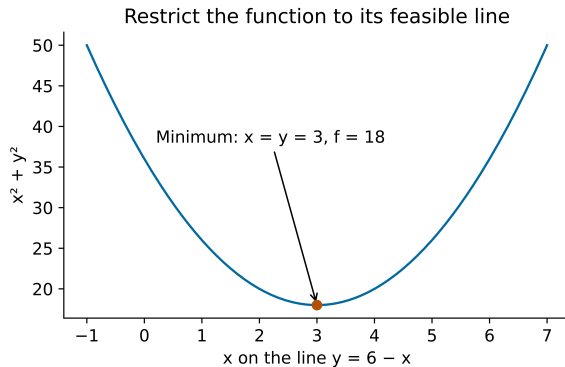
11. Worked reasoning

Worked reasoning

Minimise $x^2 + y^2$ subject to $x + y = 6$. The constraint gradient is $(1, 1)$, so $2x = \lambda$ and $2y = \lambda$. Thus $x = y$ and the constraint gives $x = y = 3$, with value 18.

Completing the square on the line gives $x^2 + (6 - x)^2 = 2(x - 3)^2 + 18$, proving a global minimum. There is no maximum on this unbounded line, despite the existence of a multiplier candidate.

Figure 12: Worked reasoning



Check the labelled values and conditions.

13. Conditions and counterexamples

Conditions and counterexamples

A Lagrange multiplier solution is a candidate, not automatically a maximum or minimum. Check regularity and the complete feasible set.

14. Try independently

Try independently

Classify the origin for $f(x, y) = x^2 - y^2$ and for $g(x, y) = x^4 + y^4$. State what a zero Hessian determinant permits you to conclude.

15. Reasoning and check

Reasoning and check

For f the Hessian is $\text{diag}(2, -2)$ with determinant -4 ; the origin is a saddle. Along the axes the values have opposite signs. For g the Hessian at zero is zero, so the second-derivative test is inconclusive. Nevertheless $x^4 + y^4 \geq 0$, with equality only at the origin; it is a strict global minimum. An inconclusive test is not a classification of the point.

16. Try independently

Try independently

Find all extrema of $f(x, y) = xy$ subject to $x^2 + y^2 = 2$. Compare with minimising $x^2 + y^2$ on the line $x + y = 4$, including whether maxima exist.

17. Reasoning and check

Reasoning and check

The circle is compact and its constraint gradient $(2x, 2y)$ never vanishes there. Multipliers give $y = 2\lambda x$, $x = 2\lambda y$. Neither coordinate can be zero at a stationary point; hence $y = \pm x$. The four candidates $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$ give global maximum 1 and minimum -1. The bound $2|xy| \leq x^2 + y^2 = 2$ checks completeness. On the line, $x^2 + (4 - x)^2 = 2(x - 2)^2 + 8$, so the minimum is 8 at $(2, 2)$. There is no maximum because the expression is unbounded. Multiplier candidates alone do not establish compactness.

Check yourself

Reconstruct the method without looking back

Find all extrema of $f(x, y) = xy$ subject to $x^2 + y^2 = 2$. Compare with minimising $x^2 + y^2$ on the line $x + y = 4$, including whether maxima exist.