

C.3 · Differential equations and initial conditions

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Differentiation, integration, exponentials and quadratic equations.

3. Scope and prerequisites

Scope and prerequisites

Solve separable first-order equations Solve constant-coefficient second-order equations
Use initial conditions and uniqueness conditions

4. Scope and prerequisites

Scope and prerequisites

initial condition : A value of the solution or derivative specified at a starting point.

5. Scope and prerequisites

Scope and prerequisites

characteristic root : A root of the polynomial governing exponential solutions.

6. Choose and justify a method

Choose and justify a method

For $y' = ky$, separation and integration give $y = Ce^{kt}$; recover the zero solution if division by y was used. An initial value determines C .

7. Choose and justify a method

Choose and justify a method

A linear second-order constant-coefficient equation uses a characteristic polynomial. Distinct roots yield exponentials; a repeated root requires $(C_1 + C_2 t)e^{rt}$.

8. Choose and justify a method

Choose and justify a method

Complex roots $\alpha \pm i\beta$ give $e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$. The real motion includes both amplitude and phase information.

9. Choose and justify a method

Choose and justify a method

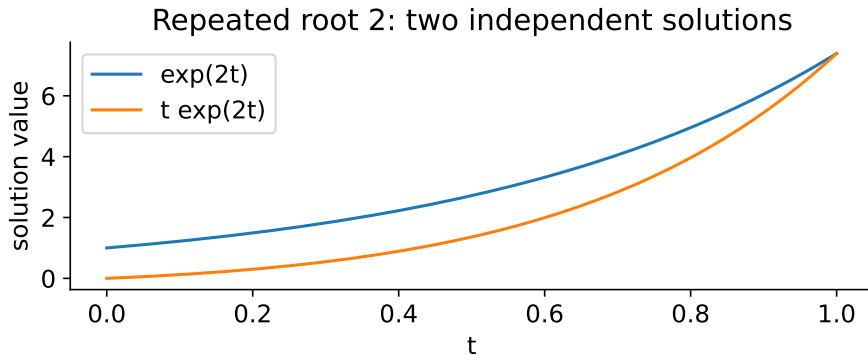
Existence and uniqueness depend on hypotheses near the initial point. A singular coefficient or a failure of local Lipschitz behaviour can defeat the familiar uniqueness conclusion.

10. Worked reasoning

Worked reasoning

Solve $y'' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 4$. The characteristic roots are $\pm 2i$, so $y = A \cos 2t + B \sin 2t$. The first condition gives $A = 3$; differentiating gives $y'(0) = 2B = 4$, so $B = 2$.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

A repeated characteristic root needs the factor t ; two copies of the same exponential are not independent solutions.

13. Try independently

Try independently

Solve $y'' - 4y' + 4y = 0$ with $y(0) = 1$, $y'(0) = 0$. Check both initial conditions and the equation.

14. Reasoning and check

Reasoning and check

The characteristic polynomial is $(r - 2)^2$. Independent solutions are e^{2t} and te^{2t} , hence $y = (C_1 + C_2 t)e^{2t}$. The initial value gives $C_1 = 1$; the derivative at zero gives $2C_1 + C_2 = 0$, so $C_2 = -2$.

$y(t) = (1 - 2t)e^{2t}$, $y'(t) = -4te^{2t}$, $y''(t) = (-4 - 8t)e^{2t}$. Substitution gives $y'' - 4y' + 4y = 0$, and the two initial values hold. Two copies of e^{2t} would not span the solution space.

15. Try independently

Try independently

Show that the initial-value problem $y' = 2\sqrt{|y|}$, $y(0) = 0$ has more than one solution for $t \geq 0$. Identify the missing standard uniqueness hypothesis.

16. Reasoning and check

Reasoning and check

One solution is identically zero. For any $a \geq 0$, set $y_a(t) = 0$ for $0 \leq t \leq a$ and $y_a(t) = (t - a)^2$ for $t > a$. The derivative is zero on the first interval and $2(t - a)$ on the second; both sides have derivative zero at a . Therefore y_a is differentiable and satisfies the equation and initial value. The right side is continuous but is not locally Lipschitz in y at zero: $|2\sqrt{h} - 0|/|h| = 2/\sqrt{h}$ is unbounded. Existence does not imply uniqueness, and division by $\sqrt{y}$ would lose the waiting-time solutions.

Check yourself

Reconstruct the method without looking back

Show that the initial-value problem $y' = 2\sqrt{|y|}$, $y(0) = 0$ has more than one solution for $t \geq 0$. Identify the missing standard uniqueness hypothesis.