

C.1 · Single-variable calculus and applications

Original classroom teaching; not an official form or score predictor

1. Scope and prerequisites

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

2. Scope and prerequisites

Scope and prerequisites

Prerequisites: Exact algebra, trigonometry in radians, functions and inequalities.

3. Scope and prerequisites

Scope and prerequisites

Use limits, continuity and differentiability Apply derivatives, integrals and the fundamental theorem Recognise Riemann sums and distinguish them from infinite series

4. Scope and prerequisites

Scope and prerequisites

continuity : Agreement of a function value with its limit.

5. Scope and prerequisites

Scope and prerequisites

convergence : Approach to a finite limiting value.

6. Choose and justify a method

Choose and justify a method

A limit describes values near a point, without requiring the function to be defined there. Continuity adds the requirement that the function value exists and agrees with the limit. Algebraic cancellation is valid only away from the cancelled zero, but can reveal a removable limit. One-sided limits must agree for a two-sided limit. For quotient limits, check the denominator and hypotheses before applying a rule; $0/0$ is an indeterminate form, not an answer.

7. Choose and justify a method

Choose and justify a method

A derivative is a limit of difference quotients and implies continuity; the converse fails, as $|x|$ at zero shows. Product, quotient and chain rules describe different structures: the derivative of $f(g(x))$ is $f'(g(x))g'(x)$, not a product of unrelated values. Differentiating a composition a second time generally produces two terms. Check domains, nonzero denominators and differentiability assumptions before using a symbolic expression as a derivative.

8. Choose and justify a method

Choose and justify a method

The fundamental theorem says that the derivative of $\int$ from a to x of a continuous integrand $f(t)$ is $f(x)$. With a variable upper bound $h(x)$, multiply by $h'(x)$; with two variable bounds, subtract the corresponding lower-bound contribution. A definite integral also arises as a limit of Riemann sums. Rewrite the sum as $(1/n)\sum f(k/n)$ before identifying the integral on $[0,1]$, rather than treating n -dependent terms as constants.

9. Choose and justify a method

Choose and justify a method

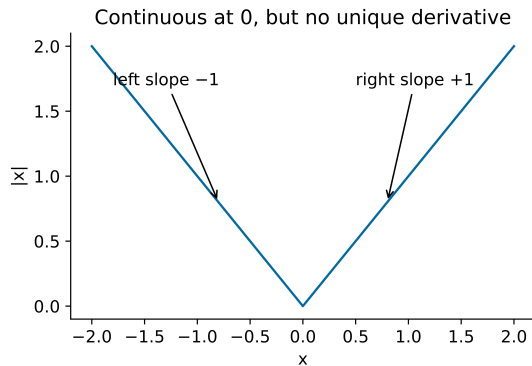
For example $\sum$ from $k=1$ to n of $n/(n^2+k^2)$ equals $(1/n)\sum 1/(1+(k/n)^2)$, tending to $\int_0^1 1/(1+t^2)dt=\pi/4$. This limit uses continuity and the partition width $1/n$. A series over an unbounded number of terms is a different limit: terms tending to zero do not alone guarantee convergence. For $\sum 1/k$ the partial sums diverge; compare, estimate or apply a valid series test instead of using the necessary term condition as sufficient.

10. Worked reasoning

Worked reasoning

For $F(x) = \int_0^{x^2} e^t dt$, the fundamental theorem and chain rule give $F'(x) = e^{x^2} \cdot 2x$. At $x=1$, $F'(1) = 2e$. The upper limit is x^2 , so omitting $2x$ misses its rate of change.

Figure 11: Worked reasoning



Check the labelled values and conditions.

12. Conditions and counterexamples

Conditions and counterexamples

A-level calculus is useful prerequisite material but does not cover the undergraduate analysis and applications tested here.

13. Try independently

Try independently

Let $F(x) = \int_x^{x^2} (1 + t^2) dt$. Find $F'(x)$ and $F'(2)$. State why the theorem applies.

14. Reasoning and check

Reasoning and check

The integrand is continuous on the real line. The fundamental theorem and chain rule apply at both bounds: $F'(x) = (1 + x^4)2x - (1 + x^2)$.

$F'(2) = (1 + 2^4)2 \cdot 2 - (1 + 2^2) = 63$. Independently, an antiderivative is $t + t^3/3$.

Thus $F = x^2 + x^6/3 - x - x^3/3$, with the same derivative.

15. Try independently

Try independently

Find $\lim_{n \rightarrow \infty} \sum_{k=1}^n k/(n^2 + k^2)$. Explain why this is a Riemann-sum limit. Then decide whether $\sum_{k=1}^{\infty} 1/k$ converges.

16. Reasoning and check

Reasoning and check

Rewrite each term as $\frac{1}{n} \frac{k/n}{1+(k/n)^2}$. The continuous function $f(t) = t/(1+t^2)$ on $[0, 1]$ and mesh $1/n$ give $L = \int_0^1 \frac{t}{1+t^2} dt = \frac{1}{2} \ln 2$. The harmonic series diverges: the block from $2^{j-1} + 1$ through 2^j contains 2^{j-1} terms at least $1/2^j$, so each block adds at least $1/2$. Terms tending to zero is necessary but insufficient. The finite-sum integrand here depends on the scaled index; it is not the harmonic series.

Check yourself

Reconstruct the method without looking back

Find $\lim_{n \rightarrow \infty} \sum_{k=1}^n k/(n^2 + k^2)$. Explain why this is a Riemann-sum limit. Then decide whether $\sum_{k=1}^{\infty} 1/k$ converges.