

C.12 • Related rates and removable quotient limits

GRE • GRE Subject Test • GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Chain rule, circular cross-sections and limits.

- Translate a geometric rate into a derivative of the relevant quantity
- Derive and differentiate a spherical-cap volume formula
- Separate continuous quotient extension from differentiable extension

related rate : A rate obtained by differentiating the relationship between changing quantities.

removable limit : A finite nearby limit used to fill in a missing function value continuously.

Choose and justify a method

For a quantity V depending on a changing depth $h(t)$, the chain rule gives $dV/dt = V'(h) dh/dt$. A draining tank has $dh/dt < 0$, so signed dV/dt is negative; an outflow magnitude is its negative. Draw the geometry and name the instantaneous depth before substituting numbers. Do not treat the rate as a static volume divided by elapsed time unless the situation actually specifies a constant average rate.

A spherical tank of radius R has cross-section radius squared $R^2 - (R-h)^2 = 2Rh - h^2$ at depth h from the bottom. Integrating these circular areas gives the cap volume $V(h) = \pi(Rh^2 - h^3/3)$, for $0 \leq h \leq 2R$. Its derivative $\pi(2Rh - h^2)$ is the current cross-sectional area. At $R=3, h=1, dh/dt = -1/4$, signed volume rate is $-5\pi/4$ and outflow magnitude $5\pi/4$. The formula works for both shallow and deep caps within the stated range.

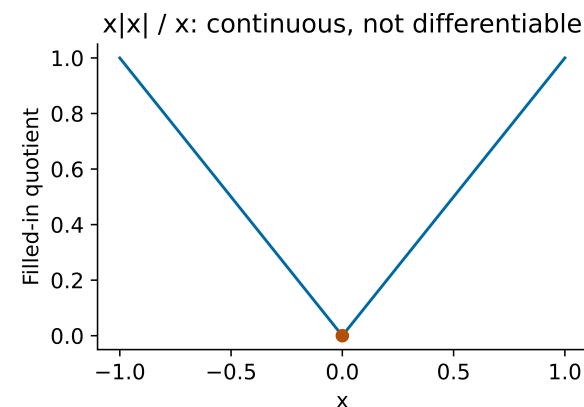
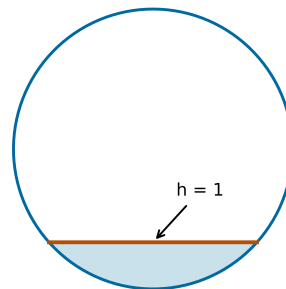
If f and g are continuously differentiable near zero, $f(0)=g(0)=0$ and $g'(0) \neq 0$, then $f(x)/g(x)$ tends to $f'(0)/g'(0)$. This follows from $f(x) = f'(0)x + o(x)$ and $g(x) = g'(0)x + o(x)$; continuity of g' keeps the quotient defined nearby except at zero. Filling in this limit gives a continuous extension. The argument does not require $f'(0)$ nonzero, and it does not prove the extended quotient differentiable.

For $f(x) = x|x|$ and $g(x) = x$, both are continuously differentiable, $f(0)=g(0)=0$ and $g'(0)=1$. Their quotient for $x \neq 0$ is $|x|$, which extends continuously at zero but has unequal one-sided derivatives there. For $(f^2 - f)/(2g - g^3)$, factor to $(f/g)(f-1)/(2-g^2)$; its removable limit is $-f'(0)/(2g'(0))$. Additional factors approach -1 and 2 . Distinguish the limit of the quotient from the derivative of its extension.

Worked reasoning

For a sphere with $R=3$ and depth $h=1$, cross-sectional area is $\pi(6-1)=5\pi$. A depth decrease of $1/4$ per time unit therefore removes volume at magnitude $5\pi/4$ per time unit. For $f(x)=\sin(2x), g(x)=3x$, the quotient extends at zero with value $2/3$. This limit says nothing about the size of a tank and should not be substituted as a derivative rate without identifying the dependent quantities.

$R = 3$; section area at $h = 1$ is 5π



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Use instantaneous cross-section area, not total tank surface area. Report a signed volume change or a positive outflow as requested. A finite removable quotient limit establishes continuity, not differentiability of the filled-in quotient.

Guided application

A spherical tank has radius 2 m and water depth h measured from its bottom. Derive $V(h)$. At $h=1$ m and $dh/dt = -0.1$ m/min, find signed volume change and outflow magnitude.

■ Worked solution

At depth s , the cross-sectional radius obeys $r(s)^2 = 2Rs - s^2$. Integrating disks gives $V(h) = \pi(Rh^2 - h^3/3)$ for $0 \leq h \leq 2R$.

$$\frac{dV}{dt} = \pi(2Rh - h^2) \frac{dh}{dt}.$$

$$\frac{dV}{dt} = \pi[2(2 \text{ m})(1 \text{ m}) - (1 \text{ m})^2](-0.1 \text{ m/min}) = -0.3\pi \text{ m}^3/\text{min}.$$

The outflow magnitude is $0.3\pi \text{ m}^3/\text{min}$. The sign describes loss of stored water, not negative physical outflow.

Independent transfer

Suppose f, g are continuously differentiable near zero, $f(0) = g(0) = 0$, $f'(0) = 4$, $g'(0) = 2$. Find the removable limit of $(f^2 - f)/(2g - g^3)$. Do these assumptions imply differentiability of the extended quotient? Give a counterexample to that general implication.

■ Check after attempting

Since $g'(0)$ is nonzero, the denominator is nonzero near zero except at zero. Factor the expression as $(f/g)(f - 1)/(2 - g^2)$. Its limit is $(4/2)(-1)/2 = -1$. First-order differentiability gives $f/g \rightarrow f'(0)/g'(0)$; it does not give second-order control. For a counterexample preserving these derivatives, take $f(x) = 4x + x|x|$ and $g(x) = 2x$. They are continuously differentiable. For nonzero x the quotient becomes $(2 + |x|/2)(-1 + 4x + x|x|)/(2 - 4x^2)$. Its expansion is $-1 + 4x - |x|/4 + O(x^2)$, giving different one-sided derivatives. Thus the extension is continuous but not differentiable.