

T.11 • Weak compositions, loop invariants and flowchart tracing

GRE • GRE Subject Test • GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Binomial coefficients, complements and assignment tracing.

- Count indistinguishable allocations with nonnegative or positive constraints
- Trace a flowchart in statement order with correct reset behaviour
- Use a loop invariant and a progress measure to justify an algorithm

weak composition : An ordered allocation of a total into nonnegative integer parts.

loop invariant : A property preserved before and after each iteration of a loop.

Choose and justify a method

The number of nonnegative integer solutions of $x_1 + \cdots + x_k = n$ is $C(n+k-1, k-1)$, for $n \geq 0$ and $k \geq 1$. Represent n identical stars separated into k named boxes by $k-1$ bars; adjacent bars and end bars allow zero entries. Boxes are distinguishable, objects are not. Counting distinct tokens would instead assign each token independently, giving k^n before other constraints. Decide the model from what the objects and recipients represent, not from a familiar-looking binomial coefficient.

For positive allocations $x_i \geq 1$, put $z_i = x_i - 1$; the remaining total is $n-k$, giving $C(n-1, k-1)$ when $n \geq k$. More general lower bounds $x_i \geq a_i$ are handled by subtracting each a_i . At least one zero is the complement of every entry positive: subtract $C(n-1, k-1)$ from the nonnegative total. Overlapping cases such as exactly two empty boxes need a different count; subtracting each empty-box case independently double-counts their intersections.

Trace a flowchart one executed statement at a time, recording variables in a table. Distinguish assignment from a mathematical equality test; the right side of an assignment uses the old value before replacement. Record which branch executes next and which variables reset. A new outer-loop candidate may reset an inner sum without resetting the candidate itself. For a process that adds 1, 3, 5, ... to S from zero, after m additions $S = m^2$ and the next addend is $2m+1$.

A loop invariant is true before and after each iteration. The square-sum invariant holds initially at $m=0$ and is preserved because $m^2 + (2m+1) = (m+1)^2$. An invariant alone does not prove termination. To test whether an integer $N \geq 0$ is square by adding odd

numbers until $S \geq N$, S grows without bound; equivalently m increases and must reach a bound such as N . At exit, equality means square and overshoot means nonsquare. A printed value must be reached on an actual execution path; a plausible numerical pattern alone is not a trace.

Worked reasoning

Distribute 11 identical tokens among 4 named boxes. Nonnegative allocations number $C(14,3)=364$; all-positive allocations number $C(10,3)=120$. Therefore 244 allocations have at least one empty box. For the odd-sum test with $N=10$, S progresses 0,1,4,9,16; it overshoots and rejects 10. For $N=9$ it exits at $S=9$ after three additions. The next addends are 1,3,5,7,9; sum and next-addend columns must not be confused.

Odd-sum loop: $S = m^2$, $B = 2m + 1$ before the next addition

Iterations m	Sum S	Next addend B
0	0	1
1	1	3
2	4	5
3	9	7
4	16	9

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Identical objects and distinct objects use different models. At least one empty box is not the same as exactly one. In a flowchart, preserve update order and reset only variables that the executed path actually resets.

Guided application

Count nonnegative solutions of $a + b + c = 8$ with $a \geq 2$. How many also have $b, c \geq 1$? How many of the first set have at least one of b, c equal to zero?

■ Worked solution

Put $u = a - 2$. Then $u + b + c = 6$ has $\binom{8}{2} = 28$ nonnegative solutions. Requiring b, c positive and subtracting one from each gives total four in three nonnegative variables, counted by $\binom{6}{2} = 15$. The complement count is $28 - 15 = 13$. Directly $b=0$ gives seven, $c=0$ gives seven, and both zero is counted twice, giving $7 + 7 - 1 = 13$.

Independent transfer

Trace this algorithm for $N=15$ and $N=16$: initialise $S=0$, $k=1$; while $S < N$, execute $S=S+k$, then $k=k+2$; report whether $S=N$. State an invariant, and justify termination for integer $N \geq 0$.

■ Check after attempting

After m iterations, $S = m^2$ and $k = 2m + 1$. The trace pairs (S, k) are $(0, 1), (1, 3), (4, 5), (9, 7), (16, 9)$. Thus 15 is rejected by overshoot and 16 accepted. The invariant holds initially and is preserved since $m^2 + (2m + 1) = (m + 1)^2$. The sum grows without bound, so it eventually reaches or exceeds N ; for example $m=N$ suffices when N is positive. For $N=0$ the loop is skipped and equality holds. An invariant alone would not establish termination.