

# T.9 · Similarity, scaling and conic distance loci

GRE · GRE Subject Test · GRE Mathematics

## Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Similar triangles, distance formula and quadratic equations.

- Match triangle vertices by equal angles before forming a side ratio
- Apply length, area and volume scale factors to geometric changes
- Identify conic distance conditions and retain a signed hyperbola branch

**similarity** : Equality of corresponding angles and a common ratio of corresponding lengths.

**focus** : A fixed point used in a conic distance definition.

## Choose and justify a method

Triangles are similar when their corresponding angles agree; their corresponding side ratios are then equal. Write the vertex correspondence explicitly. If triangle APQ has angle A equal to angle A of ABC and angle P equal to angle C, its ordered correspondence is  $A \leftrightarrow A, P \leftrightarrow C, Q \leftrightarrow B$ . Thus  $AP/AC=AQ/AB=PQ/CB$ . The side AP lies on AB but corresponds to AC; matching by where a side lies or by the letter P can give the wrong ratio.

With a common length scale factor  $k>0$ , lengths multiply by  $k$ , areas by  $k^2$  and volumes by  $k^3$ . A 30% length increase means  $k=1.30$ , hence an area increase of  $1.30^2-1=0.69$ , or 69%; it is not 60%. For similar solids with a volume ratio of 27, the length ratio is 3 and the surface-area ratio 9. Percentage changes and absolute differences are different quantities; write the new-to-old ratio before converting it to a percent.

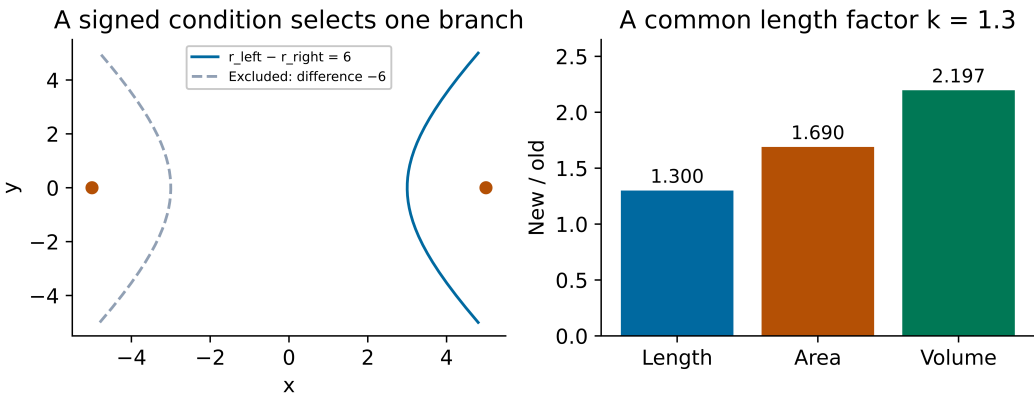
A circle fixes distance to one point. An ellipse fixes the sum of distances to two foci. A hyperbola fixes the absolute difference of those distances; a parabola fixes equality of distance to a focus and to a directrix. These are definitions of point sets, not sketches to memorise. For foci  $(-c,0)$  and  $(c,0)$ , an ellipse uses a sum  $2a$  with  $a>c$ . A nondegenerate hyperbola uses an absolute difference  $2a$  with  $0<a<c$ ; a signed difference distinguishes its two branches.

For the hyperbola with horizontal transverse axis,  $c^2=a^2+b^2$  and the equation is  $x^2/a^2-y^2/b^2=1$ . Let  $r_A$  be distance to  $(-c,0)$  and  $r_B$  distance to  $(c,0)$ . The condition  $r_A-r_B=2a>0$  selects the right branch  $x\geq a$ ; the negative condition selects the left. Squaring can lose this sign, so check the original distance condition after obtaining the equation. Difference zero gives the perpendicular bisector rather than

a hyperbola; a difference exceeding the focal separation is impossible by the triangle inequality.

## Worked reasoning

Let  $AB=10, AC=15$  and  $BC=20$ . A point P on AB has  $AP=6$ , and Q on AC makes angle APQ equal to angle ACB. Then  $AP/AC=6/15=2/5$ , so  $PQ=8$  and  $AQ=4$ . Separately, foci at  $(-5,0),(5,0)$  and distance-to-left minus distance-to-right equal to 6 give  $a=3, b^2=25-9=16$ :  $x^2/9-y^2/16=1$  with  $x\geq 3$ . The point  $(-3,0)$  satisfies the squared equation but fails the signed condition.



*Original course illustration; its values belong to the worked example, not the later practice.*

## Conditions and counterexamples

Match angles before sides. Area percentages use the square of the length factor. A squared distance equation may describe both branches even when the original problem asks for only one.

## Guided application

Similar solids have surface-area ratio 9:25. Find their corresponding length and volume ratios. Explain why matching vertices matters before forming a triangle ratio.

### ■ Worked solution

Taking positive square roots gives length ratio 3:5. Cubing gives volume ratio 27:125. The area ratio is not itself a length scale. Corresponding vertices must match equal angles; otherwise ratios can compare unrelated sides. Similarity is an assumption here; area ratios alone do not prove that arbitrary solids are similar.

## Independent transfer

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For foci  $A = (-5, 0)$  and  $B = (5, 0)$ , find the locus  $|PA| - |PB| = 6$ . Derive its standard equation, choose its branch and check the vertex in the unsquared relation. What happens if the requested difference is 12?

### ■ Check after attempting

Write  $r_A = \sqrt{(x+5)^2 + y^2}$  and  $r_B = \sqrt{(x-5)^2 + y^2}$ . The original condition is  $r_A - r_B = 6$ . Since  $r_A^2 - r_B^2 = 20x$ , multiplying the distance difference by their sum gives  $r_A + r_B = 10x/3$ . Hence  $r_B = 5x/3 - 3$ , which must be nonnegative. Squaring this equality gives  $(x-5)^2 + y^2 = (5x/3 - 3)^2$ . Expanding and collecting gives  $16x^2/9 - y^2 = 16$ . Thus the half focal distance is  $c=5$ , half difference is  $a=3$ , and the hyperbola is  $x^2/9 - y^2/16 = 1$ , using  $b^2 = c^2 - a^2 = 16$ . The original positive difference selects the right branch  $x \geq 3$ . At  $(3, 0)$  the distances are 8 and 2, difference six; at  $(-3, 0)$  the difference is negative six. Squaring alone retains this invalid branch. A difference of 12 exceeds  $|AB| = 10$  and is impossible by the reverse triangle inequality.