

T.9 · Similarity, scaling and conic distance loci

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Similar triangles, distance formula and quadratic equations.

- Match triangle vertices by equal angles before forming a side ratio
- Apply length, area and volume scale factors to geometric changes
- Identify conic distance conditions and retain a signed hyperbola branch

similarity : Equality of corresponding angles and a common ratio of corresponding lengths.

focus : A fixed point used in a conic distance definition.

Choose and justify a method

Triangles are similar when their corresponding angles agree; their corresponding side ratios are then equal. Write the vertex correspondence explicitly. If triangle APQ has angle A equal to angle A of ABC and angle P equal to angle C, its ordered correspondence is $A \leftrightarrow A$, $P \leftrightarrow C$, $Q \leftrightarrow B$. Thus $AP/AC=AQ/AB=PQ/CB$. The side AP lies on AB but corresponds to AC; matching by where a side lies or by the letter P can give the wrong ratio.

With a common length scale factor $k>0$, lengths multiply by k , areas by k^2 and volumes by k^3 . A 30% length increase means $k=1.30$, hence an area increase of $1.30^2-1=0.69$, or 69%; it is not 60%. For similar solids with a volume ratio of 27, the length ratio is 3 and the surface-area ratio 9. Percentage changes and absolute differences are different quantities; write the new-to-old ratio before converting it to a percent.

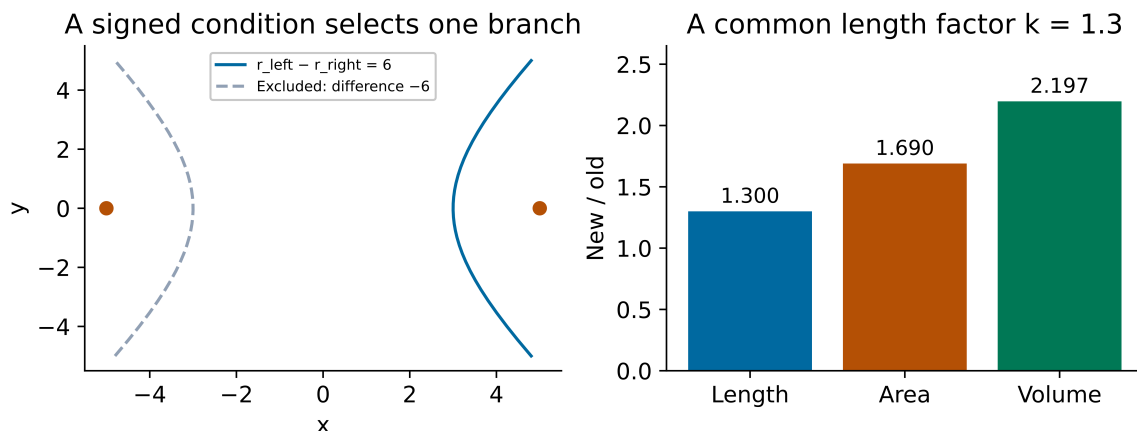
A circle fixes distance to one point. An ellipse fixes the sum of distances to two foci. A hyperbola fixes the absolute difference of those distances; a parabola fixes equality of distance to a focus and to a directrix. These are definitions of point sets, not sketches to memorise. For foci $(-c,0)$ and $(c,0)$, an ellipse uses a sum $2a$ with $a>c$. A nondegenerate hyperbola uses an absolute difference $2a$ with $0<a<c$; a signed difference distinguishes its two branches.

For the hyperbola with horizontal transverse axis, $c^2=a^2+b^2$ and the equation is $x^2/a^2-y^2/b^2=1$. Let r_A be distance to $(-c,0)$ and r_B distance to $(c,0)$. The condition $r_A-r_B=2a>0$ selects the right branch $x\geq a$; the negative condition selects the left. Squaring can lose this sign, so check the original distance condition after obtaining the equation. Difference zero gives the perpendicular bisector rather than

a hyperbola; a difference exceeding the focal separation is impossible by the triangle inequality.

Worked reasoning

Let $AB=10$, $AC=15$ and $BC=20$. A point P on AB has $AP=6$, and Q on AC makes angle APQ equal to angle ACB . Then $AP/AC=6/15=2/5$, so $PQ=8$ and $AQ=4$. Separately, foci at $(-5,0)$, $(5,0)$ and distance-to-left minus distance-to-right equal to 6 give $a=3$, $b^2=25-9=16$: $x^2/9-y^2/16=1$ with $x \geq 3$. The point $(-3,0)$ satisfies the squared equation but fails the signed condition.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Match angles before sides. Area percentages use the square of the length factor. A squared distance equation may describe both branches even when the original problem asks for only one.

Guided application

Similar solids have surface-area ratio 9:25. Find their corresponding length and volume ratios. Explain why matching vertices matters before forming a triangle ratio.

■ Worked solution

Taking positive square roots gives length ratio 3:5. Cubing gives volume ratio 27:125. The area ratio is not itself a length scale. Corresponding vertices must match equal angles; otherwise ratios can compare unrelated sides. Similarity is an assumption here; area ratios alone do not prove that arbitrary solids are similar.

Independent transfer

For foci $A = (-5, 0)$ and $B = (5, 0)$, find the locus $|PA| - |PB| = 6$. Derive its standard equation, choose its branch and check the vertex in the unsquared relation. What happens if the requested difference is 12?

■ Check after attempting

Write $r_A = \sqrt{(x+5)^2 + y^2}$ and $r_B = \sqrt{(x-5)^2 + y^2}$. The original condition is $r_A - r_B = 6$. Since $r_A^2 - r_B^2 = 20x$, multiplying the distance difference by their sum gives $r_A + r_B = 10x/3$. Hence $r_B = 5x/3 - 3$, which must be nonnegative. Squaring this equality gives $(x-5)^2 + y^2 = (5x/3 - 3)^2$. Expanding and collecting gives $16x^2/9 - y^2 = 16$. Thus the half focal distance is $c=5$, half difference is $a=3$, and the hyperbola is $x^2/9 - y^2/16 = 1$, using $b^2 = c^2 - a^2 = 16$. The original positive difference selects the right branch $x \geq 3$. At $(3, 0)$ the distances are 8 and 2, difference six; at $(-3, 0)$ the difference is negative six. Squaring alone retains this invalid branch. A difference of 12 exceeds $|AB| = 10$ and is impossible by the reverse triangle inequality.