

T.6 · Functions, inverse branches and composition

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Function domains, quadratic equations and composition.

- Distinguish injectivity and surjectivity using the stated domain and codomain
- Choose and verify an inverse branch by composing in both directions
- Trace repeated function composition while preserving the domain

bijection : A function that is both injective and surjective between its specified sets.

involution : A function whose composition with itself is the identity on its domain.

Choose and justify a method

A function assigns exactly one output to each allowed input. Injective means equal outputs force equal inputs; surjective means every element of the stated codomain is reached. The map $x \mapsto x^2$ from $\mathbb{R}$ to $[0, \infty)$ is surjective but not injective. From $[0, \infty)$ to $[0, \infty)$ it is both, hence bijective. From $[0, \infty)$ to $\mathbb{R}$ it remains injective but fails surjectivity. Keep domain, image and codomain separate when testing each claim.

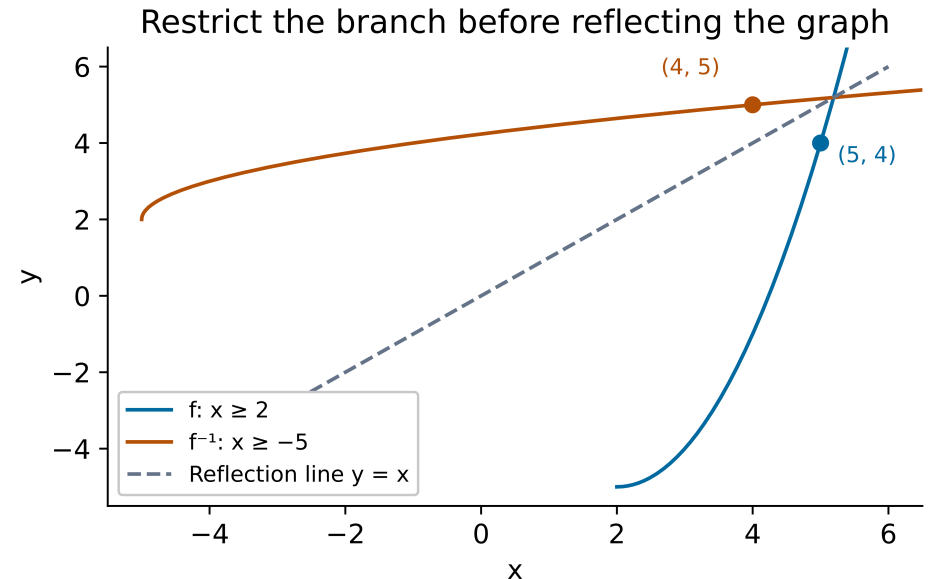
An inverse function reverses a bijection. Solve $y=f(x)$ for x , then use the original domain to choose a branch. For $f(x)=(x-2)^2-5$ on $x \geq 2$, the inverse is $2+\sqrt{y+5}$ on $y \geq -5$. The minus branch would return inputs outside the chosen domain. Check $f^{-1}(f(x))=x$ for allowed x and $f(f^{-1}(y))=y$ for allowed y ; one unchecked composition can conceal a domain error.

The inverse graph reflects the original graph across $y=x$; it does not take reciprocals of output values. A self-inverse function, or involution, satisfies $f(f(x))=x$ wherever the composition is defined. Both $-x$ on $\mathbb{R}$ and $1/x$ on $\mathbb{R}$ excluding zero are involutions. A strictly increasing involution on an interval must be the identity: if $f(x)>x$, increasingness gives $f(f(x))>f(x)>x$, and the analogous argument rules out $f(x)<x$.

Iteration f^n means repeated composition, not the power $(f(x))^n$. Calculate the first few compositions and check their domains before looking for a period. For $f(x)=1/(1-x)$ on $\mathbb{R}$ excluding 0 and 1, $f^2(x)=(x-1)/x$ and $f^3(x)=x$. The image stays in the same allowed domain. Therefore reduce an iteration count modulo three. A displayed formula equal to x after cancellation does not restore forbidden inputs.

Worked reasoning

For $f(x)=(x-2)^2-5$ with $x \geq 2$, $f(5)=4$ and $f^{-1}(4)=2+\sqrt{9}=5$. The unrestricted quadratic would have two inputs, -1 and 5 , giving output 4 . For $g(x)=1/(1-x)$, the orbit $3 \rightarrow -1/2 \rightarrow 2/3 \rightarrow 3$ has period three, so $g^8(3)=2/3$; the count concerns compositions, not eighth powers.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Changing a codomain changes surjectivity. Reflection across $y=x$ is an inverse graph; reciprocating y -values is a different operation. Do not cancel away excluded inputs.

Guided application

Restrict $f(x) = (x-1)^2 + 2$ to $x \geq 1$. State its image and inverse, including domains. Verify both inverse compositions.

■ Worked solution

The image is $[2, \infty)$. Solving $y = (x-1)^2 + 2$ with $x \geq 1$ selects $x = 1 + \sqrt{y-2}$. Thus $f^{-1}: [2, \infty) \rightarrow [1, \infty)$ has that formula. For allowed x , $1 + \sqrt{(x-1)^2} = x$ because $x-1 \geq 0$. For allowed y , $(\sqrt{y-2})^2 + 2 = y$. The negative square-root branch belongs to a different restriction, not a second inverse on this domain.

Independent transfer

Let $h(x) = 1/(1 - x)$. Determine the domain of $h \circ h \circ h$ and its formula. May the simplified expression be extended to the excluded points without changing the original composite?

■ Check after attempting

The first stage requires $x \neq 1$. The second requires $h(x) \neq 1$, which excludes $x=0$. For $x \neq 0, 1$, $h^2(x) = (x - 1)/x$. This never equals one, so the third stage adds no exclusion. Then $h^3(x) = x$ on $\mathbb{R} \setminus \{0, 1\}$. The identity formula is defined more widely, but adding 0 or 1 defines an extension, not the original composite. Each intermediate input must be permitted before cancellation.