

C.5 · Improper integrals and geometric applications

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Antiderivatives, limits, comparison and geometric integration.

- Define improper integrals by limits at each singular boundary
- Select area, volume and arc-length formulas from the geometry
- Separate convergence of integrals from signed cancellation

improper integral : An integral defined by limits at infinite or singular boundaries.

principal value : A limit using prescribed symmetric cancellation that may exist when an ordinary improper integral diverges.

Choose and justify a method

An infinite integration endpoint is replaced by a finite bound and a limit. At a singular point inside the interval, split the integral and require both one-sided integrals to converge separately. Symmetric cancellation can define a Cauchy principal value but does not prove convergence of the ordinary improper integral. For $1/x$ across zero, the two sides diverge even though symmetric cutoffs cancel.

The integral of $x^{-(p)}$ from 1 to infinity converges exactly when $p > 1$. From zero to 1 it converges exactly when $p < 1$. The same exponent behaves differently at the two boundaries. For positive integrands, comparison transfers convergence from a larger integrable function or divergence from a smaller nonintegrable one; keep the inequality direction correct. For an integral of a maximum or minimum, solve branch crossings and check which expression dominates each interval. For $\max(\sqrt{1-x^2}, x+1)$ on $[-1, 1]$, the interior crossing is zero; use the semicircle on $[-1, 0]$ and the line on $[0, 1]$. The area is $\pi/4 + 3/2$, not the integral of either branch over the entire interval.

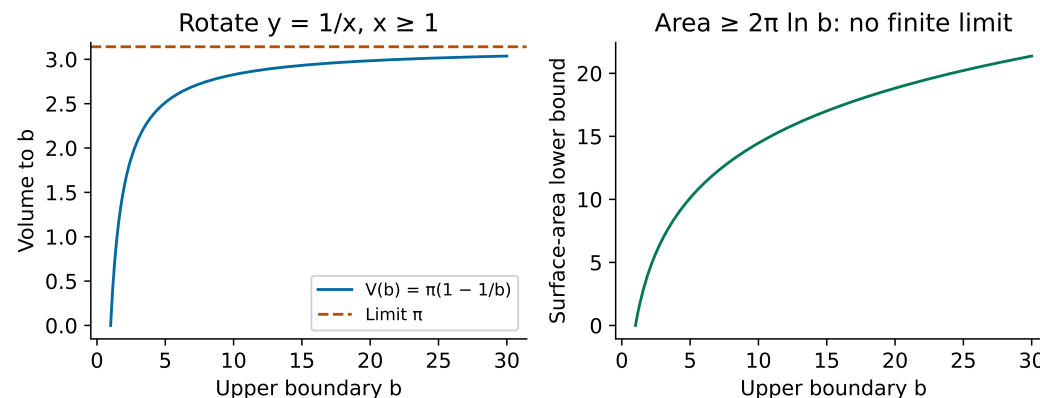
For rotation around the x-axis, disks or washers integrate $\pi(R^2 - r^2) dx$. Cylindrical shells use 2π times radius times height and integrate in the matching variable. Select the method by the geometry and verify nonnegative radii. Area between curves integrates upper minus lower, splitting where their order changes. A signed integral is not automatically geometric area.

For a differentiable plane curve $y=f(x)$, arc length is the integral of $\sqrt{1+(f'(x))^2}$ dx. A surface formed by rotating a nonnegative f around the x-axis has area integral $2\pi f \sqrt{1+(f')^2} dx$. These are different quantities from volume. If an interval is

unbounded, the geometric formula still needs a convergence test.

Worked reasoning

Rotate $y=1/x$ for $x \geq 1$ around the x-axis. The disk volume is π times the integral of $1/x^2$, so $V=\pi$. The lateral surface integrand is $2\pi(1/x)\sqrt{1+1/x^4}$, at least $2\pi/x$; its improper integral diverges. Finite volume therefore does not imply finite surface area. This comparison avoids trying to find an unnecessary antiderivative.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Never count a principal value as a convergent improper integral, or infer one geometric quantity's finiteness from another.

Guided application

For which real p does $\int_0^\infty x^{-p} dx$ converge? Compare the ordinary improper integral of $1/x$ on $[-1, 1]$ with its symmetric principal value.

■ Worked solution

Split at 1. Near zero, convergence requires $p < 1$; near infinity, it requires $p > 1$. No real p meets both, and $p=1$ diverges logarithmically at both ends. Both one-sided parts must converge independently. For $1/x$ across zero, the negative and positive parts diverge separately, so the ordinary improper integral diverges. Symmetric cutoffs cancel to zero, yielding a principal value only. The zero is not the value of the ordinary integral.

Independent transfer

Rotate the region under $y = 1/x$, $x \geq 1$, around the x-axis. Prove its volume is finite and its lateral surface area is infinite.

■ Check after attempting

Disks give $V = \pi \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx = \pi \lim(1 - 1/b) = \pi$. For surface area, $y' = -1/x^2$ and

$$S = 2\pi \int_1^\infty x^{-1} \sqrt{1 + x^{-4}} dx.$$

Its nonnegative integrand is at least $2\pi/x$, whose integral diverges. Comparison proves infinite lateral area. This includes neither an end cap nor a claim that finite volume implies finite area.