

# C.4 · Taylor expansions and power-series endpoints

GRE · GRE Subject Test · GRE Mathematics

## Scope and prerequisites

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Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Derivatives, geometric series and convergence tests.

- Construct Taylor polynomials and control approximation error
- Determine radii of convergence and test endpoints separately
- Differentiate and integrate power series within their interval of convergence

**radius of convergence** : The distance from the series centre inside which a power series converges absolutely.

**remainder** : The difference between a function and its finite approximation.

## Choose and justify a method

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The Taylor polynomial of degree  $m$  at  $a$  is the sum of  $f^{(k)}(a)(x-a)^k/k!$  for  $k$  from zero to  $m$ . The factorial belongs to each coefficient. If the next derivative is bounded in magnitude by  $M$  between  $a$  and  $x$ , the Lagrange remainder has magnitude at most  $M|x-a|^{m+1}/(m+1)!$ . Smoothness alone does not guarantee that the infinite Taylor series equals the function everywhere.

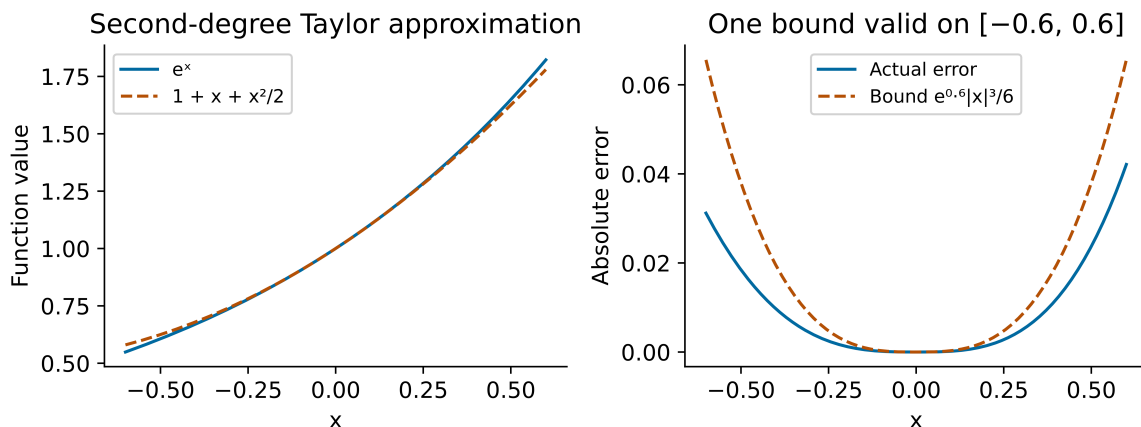
A power series  $\sum c_n(x-a)^n$  converges absolutely inside its radius  $R$  and diverges outside it. Ratio or root tests usually determine  $R$ , with possible values zero and infinity. At  $x=a-R$  and  $x=a+R$  the test often becomes inconclusive; substitute each endpoint into the original series. The two endpoint behaviours may differ.

Within the open interval of convergence, termwise differentiation and integration preserve the radius. They can change whether endpoints are included. Start from the geometric series  $1/(1-x)=\sum x^n$  for  $|x|<1$ , then integrate from zero to  $x$  to obtain  $-\ln(1-x)=\sum x^n/n$  for  $n\geq 1$ . Check the integration constant and the real logarithm domain.

Series also resolve removable limit forms. To evaluate  $(e^x-1-x)/x^2$  near zero, retain the first surviving term  $x^2/2$  rather than using only  $e^x\approx 1+x$ . An asymptotic truncation establishes the limit; a finite-interval inequality needs a separate remainder sign or magnitude argument. Do not substitute into a series outside its convergence interval.

## Worked reasoning

For  $\sum x^n/n$  with  $n \geq 1$ , the ratio test gives radius 1. At  $x=1$  it is the divergent harmonic series; at  $x=-1$  it is an alternating convergent series. Thus its real convergence interval is  $[-1, 1)$ . Differentiating inside gives  $\sum x^{n-1} = 1/(1-x)$ , which converges at neither endpoint. The radius stayed 1 while the endpoint inclusion changed.



*Original course illustration; its values belong to the worked example, not the later practice.*

## Conditions and counterexamples

A radius is not a complete interval of convergence. Endpoint tests and factorial coefficients must be checked explicitly.

## Guided application

Find the convergence interval of  $\sum_{n=1}^{\infty} x^n/n$  and of its termwise derivative. Approximate  $e^{0.1}$  by its quadratic Taylor polynomial with a justified error bound.

### ■ Worked solution

The ratio test gives radius one. At 1 the original harmonic series diverges; at -1 the alternating series converges. Thus the original interval is  $[-1, 1)$ . The derivative is  $\sum_{n=1}^{\infty} x^{n-1} = 1/(1-x)$  only for  $|x| < 1$ ; its terms do not tend to zero at either endpoint.  $P_2(0.1) = 1 + 0.1 + 0.1^2/2 = 1.105$ . On  $[0, 0.1]$ , the third derivative is  $e^x < 2$ . The Lagrange bound gives  $|R_2| \leq 2(0.1)^3/6 = 1/3000$ . The bound is deliberately conservative, not a claim of exact error.

## Independent transfer

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Evaluate  $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$ . Explain why replacing cosine by only  $1 - x^2/2$  cannot determine this limit.

### ■ Check after attempting

Taylor's theorem near zero gives  $\cos x = 1 - x^2/2 + x^4/24 + O(x^6)$ . Subtraction cancels the constant and quadratic terms, leaving  $x^4/24 + O(x^6)$ . After division the limit is  $1/24$ . A truncation with remainder merely  $o(x^2)$  gives no control after division by  $x^4$ ; retain and bound the first surviving order.