

T.5 · Open sets, compactness and connectedness

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Sets, relative open sets, continuous functions and sequences.

- Compute closure, interior and boundary in a stated space
- Apply compactness and connectedness to continuous maps
- Distinguish relative topology from the ambient Euclidean topology

relative topology : Open subsets inherited by intersecting a subspace with ambient open sets.

connectedness : Absence of a separation into disjoint nonempty relatively open subsets.

Choose and justify a method

In a metric space, an open set contains a small ball around each of its points. The interior consists of such points; the closure includes all limit points; the boundary is closure minus interior. Open and closed are not mutually exclusive labels; the empty set and the whole space are both. In $\mathbb{R}$, the set $[0,1)$ has interior $(0,1)$, closure $[0,1]$ and boundary $\{0,1\}$.

In a subspace X , an open set has the form X intersected with an ambient open set. Thus $[0,1)$ is open relative to $[0,2]$, using intersection with $(-1,1)$, although it is not open in $\mathbb{R}$. A set may also be relatively closed without being closed in the ambient space. Always state which space defines neighbourhoods and which metric is used.

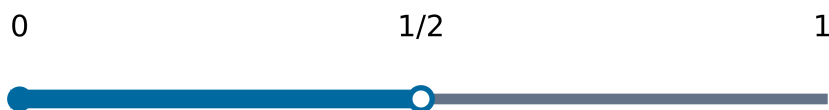
Compactness means every open cover has a finite subcover. In Euclidean $\mathbb{R}^n$, Heine–Borel makes this equivalent to closed and bounded. In a metric space, compactness is equivalent to sequential compactness; completeness and boundedness alone do not suffice in arbitrary metric spaces. A continuous image of a compact set is compact, giving attained maxima and minima for real continuous functions on a nonempty compact domain.

Connected sets cannot be separated into two disjoint nonempty relatively open parts; connected subsets of $\mathbb{R}$ are precisely intervals. A continuous image of a connected set is connected, which yields the intermediate value theorem. Path connectedness implies connectedness, but not conversely in every space. A compact set need not be connected, and a connected set need not be compact; a finite two-point set and an open interval supply the contrasting cases.

Worked reasoning

Let $X=[0,1]$ with its usual relative topology. The set $U=[0,0.5)$ equals X intersected with $(-1,0.5)$, so U is open in X . Its closure in X is $[0,0.5]$, and its boundary in X is $\{0.5\}$; zero is an interior point relative to X . For a continuous f on X with $f(0)<0<f(1)$, connectedness ensures a zero, while compactness separately ensures attained extrema.

The ambient space changes the boundary



$$X = [0, 1]; U = [0, 1/2)$$

Relative interior: U Closure in X : $[0, 1/2]$

Boundary in X : $\{1/2\}$ Boundary in $\mathbb{R}$: $\{0, 1/2\}$

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Heine–Borel’s closed-and-bounded test is a Euclidean-space theorem. Connectedness and compactness answer different questions.

Guided application

In $X = [0, 2]$ with its relative topology, find the interior, closure and boundary of $A = [0, 1)$. Is A open or closed in X ?

■ Worked solution

$A = X \cap (-1, 1)$ is open in X , so its relative interior is A . Its closure is $[0, 1]$ and its boundary is $\{1\}$. Zero is interior relative to X because a sufficiently small ball in X has no negative part. A is not closed in X : the sequence $1 - 1/n$ for $n \geq 2$ tends to the missing point 1. In the ambient real line, interior and boundary would instead be $(0, 1)$ and $\{0, 1\}$.

Independent transfer

Give a compact disconnected real set and a connected noncompact real set. Must a continuous real function on the latter attain an extreme value? Explain which property gives the intermediate value theorem.

■ Check after attempting

The finite set $\{0, 2\}$ is compact but disconnected. The interval $(0, 1)$ is connected but not compact. The continuous identity function on $(0, 1)$ attains neither a maximum nor a minimum. Connectedness makes a continuous real image an interval and therefore supplies intermediate values. Compactness instead makes a nonempty continuous real image attain its extreme values. Neither property substitutes for the other.