

# A.3 · Groups, cosets and quotient maps

GRE · GRE Subject Test · GRE Mathematics

## Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Group axioms, functions, integer divisibility and permutations.

- Verify group structure and compute element orders and subgroup indices
- Use kernels and images to identify quotient groups
- Distinguish normal subgroups from arbitrary subgroups
- Classify permutation conjugacy by cycle type

**coset** : A translate of a subgroup that forms one part of the coset partition.

**normal subgroup** : A subgroup invariant under conjugation by every group element.

## Choose and justify a method

Before computing an order, check closure, associativity, an identity and an inverse for every element under the stated operation. A subset can inherit associativity yet fail closure or omit the identity. In a finite group, the order of an element is the least positive power giving the identity. In the additive group  $\mathbb{Z}/n\mathbb{Z}$ , it is the least positive multiple giving zero; the order of residue  $a$  is  $n/\gcd(a,n)$ . Lagrange’s theorem says subgroup orders divide the group order. The converse is not a general existence theorem, and the order of a group is not the order of each element.

A left coset  $gH$  is a translate of a subgroup  $H$ . Cosets have equal size and partition the group, so the index is  $|G|/|H|$  in a finite group. In an additive group write  $g+H$ . Membership in the same coset means the difference lies in  $H$ . A coset usually is not itself a subgroup because it may omit the identity.

A homomorphism preserves the operation. Its kernel consists of elements sent to the identity, and its image consists of values actually reached. Every kernel is normal. The first isomorphism theorem identifies  $G/\ker(\phi)$  with  $\text{im}(\phi)$ ; do not replace the image with the whole codomain unless the map is onto.

Quotient multiplication is well defined only when  $H$  is normal. All subgroups of an abelian group are normal. In a nonabelian group test  $gHg^{-1}=H$ ; a subgroup of index two is normal. For permutations compose in the stated convention, here rightmost first. Disjoint cycle lengths give the permutation order by their least common multiple. Conjugation  $h\sigma h^{-1}$  relabels the elements in  $\sigma$ ’s cycles, so it preserves cycle lengths; conversely permutations with the same cycle lengths can be related by a relabelling. Thus conjugacy classes in  $S_n$  correspond to partitions of  $n$ , including fixed-point cycles. In  $S_4$  the types are  $1+1+1+1$ ,  $2+1+1$ ,  $2+2$ ,  $3+1$  and  $4$ : five classes, not one

class for each possible element order. The types  $2+1+1$  and  $2+2$  both have order two but are not conjugate.

## Worked reasoning

Define  $\phi$  from  $\mathbb{Z}/12\mathbb{Z}$  to  $\mathbb{Z}/3\mathbb{Z}$  by reducing residues modulo 3. It is onto and preserves addition. Its kernel  $H$  is  $\{0,3,6,9\}$ , so  $|H|=4$  and the index is  $12/4=3$ . The other cosets are  $\{1,4,7,10\}$  and  $\{2,5,8,11\}$ . Thus  $(\mathbb{Z}/12\mathbb{Z})/H$  is isomorphic to  $\mathbb{Z}/3\mathbb{Z}$ . The element 3 in the original group has order  $12/\gcd(3,12)=4$ , not 3.

### Reduction modulo 3: three fibres, four elements each

| Quotient class | Elements of $\mathbb{Z}/12\mathbb{Z}$ | Image in $\mathbb{Z}/3\mathbb{Z}$ |
|----------------|---------------------------------------|-----------------------------------|
| $H$            | 0, 3, 6, 9                            | 0                                 |
| $1 + H$        | 1, 4, 7, 10                           | 1                                 |
| $2 + H$        | 2, 5, 8, 11                           | 2                                 |

*Original course illustration; its values belong to the worked example, not the later practice.*

## Conditions and counterexamples

A quotient has one element per coset, not one per element of its kernel. A homomorphism need not be onto its stated codomain.

## Guided application

In the additive group  $\mathbb{Z}/18\mathbb{Z}$ , let  $H = \langle 6 \rangle$ . List  $H$  and its cosets. Determine the order of 6 and identify the quotient.

### ■ Worked solution

$H = \{0,6,12\}$  has size three. The six cosets are  $r + H = \{r, r + 6, r + 12\}$  for  $r = 0,1,2,3,4,5$ . They partition all 18 residues. The element 6 has order three, while the quotient has order six. Reduction modulo six is onto with kernel  $H$ , so  $(\mathbb{Z}/18\mathbb{Z})/H \cong \mathbb{Z}/6\mathbb{Z}$ . Normality holds because the original group is abelian.

## Independent transfer

---

In  $S_3$ , let  $H = \{e, (12)\}$ . Check whether  $H$  is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of  $(123)(45)$  in  $S_5$ .

### ■ Check after attempting

Conjugation by  $(123)$  sends  $(12)$  to  $(23)$ , which is outside  $H$ . Thus  $H$  is not normal. Left cosets still partition  $S_3$ , but multiplication of those cosets is not well defined independently of representatives. The disjoint cycles in  $(123)(45)$  have lengths three and two, so its order is  $\text{lcm}(3, 2) = 6$ . Element order and subgroup index are different quantities.