

A.3 • Groups, cosets and quotient maps

GRE • GRE Subject Test • GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Group axioms, functions, integer divisibility and permutations.

- Verify group structure and compute element orders and subgroup indices
- Use kernels and images to identify quotient groups
- Distinguish normal subgroups from arbitrary subgroups
- Classify permutation conjugacy by cycle type

coset : A translate of a subgroup that forms one part of the coset partition.

normal subgroup : A subgroup invariant under conjugation by every group element.

Choose and justify a method

Before computing an order, check closure, associativity, an identity and an inverse for every element under the stated operation. A subset can inherit associativity yet fail closure or omit the identity. In a finite group, the order of an element is the least positive power giving the identity. In the additive group $\mathbb{Z}/n\mathbb{Z}$, it is the least positive multiple giving zero; the order of residue a is $n/\gcd(a,n)$. Lagrange's theorem says subgroup orders divide the group order. The converse is not a general existence theorem, and the order of a group is not the order of each element.

A left coset gH is a translate of a subgroup H . Cosets have equal size and partition the group, so the index is $|G|/|H|$ in a finite group. In an additive group write $g+H$. Membership in the same coset means the difference lies in H . A coset usually is not itself a subgroup because it may omit the identity.

A homomorphism preserves the operation. Its kernel consists of elements sent to the identity, and its image consists of values actually reached. Every kernel is normal. The first isomorphism theorem identifies $G/\ker(\phi)$ with $\text{im}(\phi)$; do not replace the image with the whole codomain unless the map is onto.

Quotient multiplication is well defined only when H is normal. All subgroups of an abelian group are normal. In a nonabelian group test $gHg^{-1}=H$; a subgroup of index two is normal. For permutations compose in the stated convention, here rightmost first. Disjoint cycle lengths give the permutation order by their least common multiple. Conjugation $h\sigma h^{-1}$ relabels the elements in σ 's cycles, so it preserves cycle lengths; conversely permutations with the same cycle lengths can be related by a relabelling. Thus conjugacy classes in S_n correspond to partitions of n , including fixed-point cycles. In S_4 the types are $1+1+1+1$, $2+1+1$, $2+2$, $3+1$ and 4 : five classes, not one

class for each possible element order. The types 2+1+1 and 2+2 both have order two but are not conjugate.

Worked reasoning

Define ϕ from $\mathbb{Z}/12\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$ by reducing residues modulo 3. It is onto and preserves addition. Its kernel H is $\{0,3,6,9\}$, so $|H|=4$ and the index is $12/4=3$. The other cosets are $\{1,4,7,10\}$ and $\{2,5,8,11\}$. Thus $(\mathbb{Z}/12\mathbb{Z})/H$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The element 3 in the original group has order $12/\gcd(3,12)=4$, not 3.

Reduction modulo 3: three fibres, four elements each

Quotient class	Elements of $\mathbb{Z}/12\mathbb{Z}$	Image in $\mathbb{Z}/3\mathbb{Z}$
H	0, 3, 6, 9	0
$1 + H$	1, 4, 7, 10	1
$2 + H$	2, 5, 8, 11	2

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A quotient has one element per coset, not one per element of its kernel. A homomorphism need not be onto its stated codomain.

Guided application

In the additive group $\mathbb{Z}/18\mathbb{Z}$, let $H = \langle 6 \rangle$. List H and its cosets. Determine the order of 6 and identify the quotient.

■ Worked solution

$H = \{0,6,12\}$ has size three. The six cosets are $r + H = \{r, r + 6, r + 12\}$ for $r = 0, 1, 2, 3, 4, 5$. They partition all 18 residues. The element 6 has order three, while the quotient has order six. Reduction modulo six is onto with kernel H , so $(\mathbb{Z}/18\mathbb{Z})/H \cong \mathbb{Z}/6\mathbb{Z}$. Normality holds because the original group is abelian.

Independent transfer

In S_3 , let $H = \{e, (12)\}$. Check whether H is normal, and decide whether coset multiplication defines a quotient group. Separately find the order of $(123)(45)$ in S_5 .

■ Check after attempting

Conjugation by (123) sends (12) to (23) , which is outside H . Thus H is not normal. Left cosets still partition S_3 , but multiplication of those cosets is not well defined independently of representatives. The disjoint cycles in $(123)(45)$ have lengths three and two, so its order is $\text{lcm}(3, 2) = 6$. Element order and subgroup index are different quantities.