

T.2 · Discrete mathematics, probability and numerical methods

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Finite counting, elementary probability, recurrences and derivatives.

- Use counting, recurrences, graph reasoning and logic
- Calculate probabilities and distribution properties
- Apply numerical approximation and assess error

recurrence : A rule linking a sequence term to earlier terms.

mutually exclusive : Events that cannot happen together.

Choose and justify a method

Choose the counting model first: ordered selections of r distinct objects use $n(n-1)\cdots(n-r+1)$, while unordered subsets use $C(n,r)$. A complete simple graph on n vertices has one edge per unordered vertex pair, giving $n(n-1)/2$ edges; loops and multiple edges would change the model. For n lines in general position in a plane, the k th line crosses the prior $k-1$ lines in distinct points and adds k regions. The total is $1+n(n+1)/2$; parallels or triple concurrence invalidate that count.

Use complements to count at least one occurrence, and condition on the actual remaining population after a draw without replacement. For r iid outcomes chosen from n equally likely values, the probability that all are distinct is $n(n-1)\cdots(n-r+1)/n^r$ when $r \leq n$. Its complement counts repeated outcomes. For a binomial count with N independent trials and fixed success probability p , mean is Np and variance $Np(1-p)$. Identical probabilities alone do not establish independence.

A recurrence describes later values from earlier ones and needs enough initial data to determine a sequence. Separate the index from the value: $a_n = 2a_{n-1}$ with $a_0 = 3$ gives $a_n = 3 \cdot 2^n$. Graph and algorithm arguments often establish a recurrence by identifying what a new vertex or step adds. A closed formula should satisfy both the recurrence and its initial conditions; fitting a few observed terms does not prove it for every index.

Numerical approximation needs an error argument. Bisection preserves a sign-changing bracket for a continuous function and halves its width at each step; a zero may be absent if continuity fails. Newton's update is $x_{\text{new}} = x - f(x)/f'(x)$, requiring a nonzero derivative at the current point; convergence is not automatic from every starting value. An approximation's residual and its error in x are different. State the

method’s assumptions and a stopping criterion, rather than treating extra displayed decimals as accuracy.

Worked reasoning

For $f(x)=x^2-2$ and $x_0=1$, Newton’s step gives $1-(1-2)/2=1.5$. The next value is $1.5-0.25/3=1.4167$. These approximate $\sqrt{2}$, but $f'(0)=0$ makes zero an invalid starting point.

Four labelled independent draws, five equally likely symbols

Event	Count model	Count
ordered strings	$5 \times 5 \times 5 \times 5$	625
all distinct	$5 \times 4 \times 3 \times 2$	120
at least one repeat	$625 - 120$	505

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

Events with positive probabilities cannot be both independent and mutually exclusive.

Guided application

Four labelled draws are independent and uniform from five symbols, with replacement. Find the probability that at least two draws agree. Compare the number of ordered and unordered selections of four distinct symbols.

■ **Worked solution**

There are 5^4 equally likely ordered draw strings. The no-repeat strings number $5 \cdot 4 \cdot 3 \cdot 2 = 120$.

$$P(\text{repeat}) = 1 - \frac{5 \cdot 4 \cdot 3 \cdot 2}{5^4} = \frac{101}{125}.$$

Ordered distinct selections number 120; unordered subsets number $\binom{5}{4} = 5$. Each subset occurs in $4! = 24$ orders. Dividing by $4!$ is appropriate only for the unordered count, not the draw probability denominator.

Independent transfer

Apply bisection to $f(x) = x^3 - 2$ on $[1, 2]$. Give the first two retained brackets and a bound for midpoint error after ten bisections. Would a sign change alone justify bisection for $1/x$ on $[-1, 1]$?

■ Check after attempting

The polynomial is continuous; $f(1) < 0 < f(2)$. The first midpoint is 1.5 with positive value, retaining $[1, 1.5]$. The next is 1.25 with negative value, retaining $[1.25, 1.5]$. After ten halvings, width is 2^{-10} and the midpoint of that retained bracket differs from the bracketed root by at most $2^{-11} = 1/2048$. The error bound uses the bracket, not a small residual alone. For $1/x$, zero is a singularity and there is no root. Continuity on the bracket fails; opposite endpoint signs do not justify the method.