

A.1 · Linear algebra

GRE · GRE Subject Test · GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Linear equations, matrix multiplication, polynomials and complex roots.

- Relate rank, nullity and solutions of linear systems
- Analyse vector spaces and linear transformations
- Compute eigenvalues, determinants and diagonalisation conditions

rank : Dimension of the image of a linear map.

eigenvalue : A scalar satisfying $Av = \lambda v$ for a nonzero v .

Choose and justify a method

A real vector space is closed under its specified addition and scalar multiplication and satisfies the vector-space axioms. A basis is an independent spanning list: every vector has a unique coordinate representation in that list. To test independence of matrix columns, solve the homogeneous system $Ac=0$; only the zero coefficient vector means independence. A spanning set can contain redundant vectors, so spanning alone does not make a basis.

Row reduction exposes pivots and free variables without changing the solution set of a linear system. Rank is the dimension of the column image, equivalently the number of pivots. Nullity is the kernel dimension: for a map from an n -dimensional domain, $\text{rank} + \text{nullity} = n$. In a nonhomogeneous system, a zero coefficient row with nonzero right side means inconsistency; free variables give infinitely many solutions only after consistency has been established. The codomain dimension need not equal rank.

A square matrix is invertible exactly when its determinant is nonzero, its kernel is zero and its rank equals its size. These statements do not require each entry to be nonzero. Triangular determinants are products of diagonal entries, so a matrix depending on a complex variable can be singular at complex roots absent from a real-only calculation. Row swaps reverse determinant sign; adding a multiple of one row to another leaves it unchanged. Check singularity before applying an inverse formula.

An eigenvector is nonzero and satisfies $Av = \lambda v$, so eigenvalues are roots of $\det(A - \lambda I)$. Diagonalisation requires a full independent eigenvector basis over the chosen field. Distinct eigenvalues give independent eigenvectors, while repeated eigenvalues may have too small an eigenspace. A real symmetric matrix has a real orthonormal eigenbasis. A real odd-dimensional matrix has at least one real eigenvalue because its real characteristic polynomial has odd degree; that alone does not imply diagonalisation or all eigenvalues real.

Worked reasoning

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has characteristic polynomial $(1 - \lambda)^2$. Eigenvectors satisfy $y=0$, so its eigenspace has dimension one. Two independent eigenvectors are needed to diagonalise a 2×2 matrix; A is not diagonalizable.

A repeated eigenvalue can have different eigenspace dimensions

Matrix	Eigenvalue	Dimension	Diagonalizable?
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	1, twice	1	No
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1, twice	2	Yes

Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

The algebraic multiplicity of an eigenvalue is not automatically the dimension of its eigenspace.

Guided application

For $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $T(x, y, z) = (x + 2y + z, 2x + 4y + 2z)$, find bases of the kernel and image. Solve $T(v) = (3, 6)$ and $T(v) = (3, 7)$.

■ Worked solution

The second row is twice the first, so rank is one. The homogeneous equation gives $x = -2y - z$.

$$\ker T = \text{span}\{(-2, 1, 0), (-1, 0, 1)\}.$$

The two displayed vectors are independent. Every output is $(s, 2s)$, and $(1, 2)$ occurs; this is an image basis. Rank plus nullity is $1 + 2 = 3$, the domain dimension. For $(3, 6)$, all solutions are $(3 - 2a - b, a, b)$, with $a, b \in \mathbb{R}$. For $(3, 7)$ there is no solution, since $7 \neq 2 \cdot 3$.

Independent transfer

Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Determine its eigenvalues, determinant and diagonalisation over $\mathbb{R}$ and $\mathbb{C}$. Find A^n for integers $n \geq 1$ without repeated multiplication.

■ Check after attempting

The characteristic polynomial is $(2 - \lambda)^2$ and $\det A = 4$. Solving $(A - 2I)v = 0$ gives $v_2 = 0$, so the eigenspace has dimension one over either field. Two independent eigenvectors are required; the matrix is invertible but not diagonalizable over either field. Write $A = 2I + N$ where $N^2 = 0$. The commuting binomial expansion leaves only two terms:

$$A^n = 2^n I + n2^{n-1}N = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}.$$

Multiplying this expression by A gives the formula at $n + 1$, independently checking the coefficient.