

C.2 • Multivariable calculus and vector analysis

GRE • GRE Subject Test • GRE Mathematics

Scope and prerequisites

Undergraduate GRE preparation. Local objectives within the reviewed ETS scope; this is original teaching, not an official test or score predictor.

Prerequisites: Single-variable derivatives, dot products and iterated integration.

- Compute partial derivatives, gradients and directional derivatives
- Use multiple integrals and coordinate changes
- Use derivative conditions to distinguish planes from curved surfaces

gradient : Vector of partial derivatives.

Jacobian : The local area or volume scaling in a coordinate change.

Choose and justify a method

A partial derivative changes one coordinate while fixing the others. For $f=x^2+3xy$, $f_x=2x+3y$ and $f_y=3x$. The gradient collects these derivatives; if f is differentiable, the directional derivative along a unit vector $\mathbf{v}$ is $\text{grad } f \cdot \mathbf{v}$. Normalise the direction before taking this dot product. A direction vector of length two would double the answer if used without normalisation.

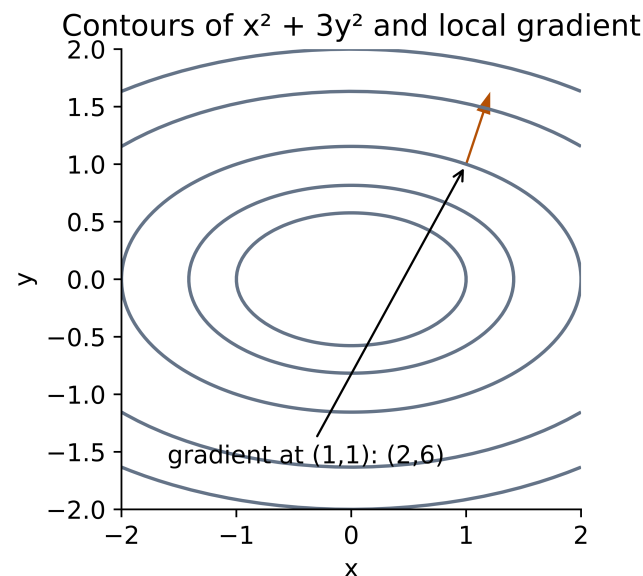
Differentiability means a valid linear approximation, not merely the existence of some partial derivatives at one point. Continuous first partials in a neighbourhood are a sufficient condition. For a composition $f(x(t), y(t))$, the chain rule gives $f_x x' + f_y y'$. Second derivatives can introduce both direct and mixed terms; a zero mixed partial alone places no restriction on the pure second partials.

A multiple integral sums contributions over a specified region. Describe the region before choosing iterated limits; nonrectangular bounds may change when the order changes. In polar coordinates area is $r \, dr \, d\theta$, not just $dr \, d\theta$. A general coordinate change uses the absolute Jacobian determinant. Sign belongs to oriented vector quantities, while area and volume scaling use a nonnegative factor.

If both first partials of a globally defined function on $\mathbb{R}^2$ are constant, $f_x=a$ and $f_y=b$, integrating successively gives $f=ax+by+c$, a plane. This cannot be inferred from parallel straight level sets alone: e^x has vertical parallel level lines but a curved graph. Nor do $f_{xy}=f_{yx}=0$ force a plane: x^2+y^2 is a counterexample. Distinguish first-derivative constancy from absent mixed dependence and from the shape of selected level sets.

Worked reasoning

For $f(x,y)=x^2+3y^2$, $\text{grad } f=(2x, 6y)$. At $(1,1)$ this is $(2,6)$. In direction $(3,4)$, the unit vector is $(3/5, 4/5)$, giving directional derivative $2 \cdot 3/5 + 6 \cdot 4/5 = 6$.



Original course illustration; its values belong to the worked example, not the later practice.

Conditions and counterexamples

A non-unit direction vector gives a scaled directional rate, not the derivative per unit distance.

Guided application

For $f(x,y) = x^2 + xy + 2y^2$, find the directional derivative at $(1, -1)$ towards $(3, 4)$. Evaluate $\iint_D (x^2 + y^2) \, dA$ for the unit disk D .

■ Worked solution

The polynomial is differentiable. Normalise the direction: $\mathbf{u} = (3/5, 4/5)$. The gradient is $(2x + y, x + 4y)$, hence $(1, -3)$ at the point.

$$D_{\mathbf{u}}f(1, -1) = \nabla f(1, -1) \cdot \mathbf{u} = 1 \cdot 3/5 - 3 \cdot 4/5 = -9/5.$$

Polar coordinates give $x^2 + y^2 = r^2$ and $dA = r \, dr \, d\theta$:

$$I = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \pi/2.$$

Independent transfer

Define $g(x, y) = xy/\sqrt{x^2 + y^2}$ away from the origin and $g(0, 0) = 0$. Both partial derivatives at the origin are zero. Is g differentiable there? Is it continuous?

■ Check after attempting

Along each axis $g = 0$, so both partials are zero. Continuity follows from $|xy| \leq (x^2 + y^2)/2$, which gives $|g(x, y)| \leq \sqrt{x^2 + y^2}/2 \rightarrow 0$. If differentiable, its linear derivative would be zero. Along $x = y = t \neq 0$,

$$\frac{|g(t, t)|}{\sqrt{t^2 + t^2}} = \frac{1}{2}.$$

The required remainder ratio does not tend to zero. Thus continuity and existing partial derivatives do not establish differentiability.