

Solutions

2.1 C. The injective map defines a metric. The image is an open bounded interval, and $\arctan n$ tends to a missing endpoint. Injectivity proves this is a metric. Continuity does not give completeness, unbounded n is Cauchy in this metric, and arctangent distance is not ordinary absolute distance.

2.2 E. Closure consists of multiples of 8. The point 16 has every neighbourhood containing 8; U_4 demonstrates why the divisor 4 does not belong. Closure requires multiples of eight. Two, three, four and six have divisor neighbourhoods missing eight; sixteen is a multiple and every neighbourhood contains eight.

2.3 Both defining maps are injective, giving positive separation; absolute distance gives symmetry and the triangle inequality. Each map is an isometry onto its real image. The cube map has image all of $\mathbb{R}$, a complete space, so d_1 is complete. The arctangent image is the open interval $(-\pi/2, \pi/2)$, which is incomplete. The sequence n is d_2 -Cauchy because $\arctan n$ tends to $\pi/2$, but no real point has that image, so it has no d_2 -limit. Bounded metric values do not imply completeness.

3.1 Every x belongs to U_x . An intersection is $U_{\gcd(k,l)}$ if the gcd is at least two, and is empty for gcd one. Thus a point in an intersection has a basic neighbourhood within it. The smallest basic neighbourhood of x is U_x , contained in every U_k that contains x . It meets $\{6\}$ exactly when $6 \mid x$. Therefore the closure is $\{6, 12, 18, \dots\}$, the multiples of six. The point six is not interior to its singleton because U_6 also contains two and three. No other point is in that singleton, so its interior is empty. This closure calculation uses all basic neighbourhoods, not Euclidean intuition.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.