

Solutions

2.1 D. $y_1 = -1$ and the second row is $2y_1 + y_2 = -1$. Therefore $y_2 = 1$. Substituting $y_1 = -1$ in $2y_1 + y_2 = -1$ gives $y_2 = 1$. Minus three, minus one, zero and twelve fail that second row.

2.2 C. Nullity B contributes 3, and $\text{im } B \cap \ker A$ has dimension from 0 to 2 in this ambient space. Nullity cannot be below three because $\ker B$ is included. It cannot exceed five because at most two new kernel dimensions arise from $\ker A$. Always five ignores images disjoint from $\ker A$.

2.3 First solve $Ly = b$: $y_1 = 5$, $2y_1 + y_2 = 12$, so $y_2 = 2$. Then $Ux = y$ gives $x_2 = 1$ and $3x_1 + 1 = 5$, hence $x_1 = 4/3$. The product matrix is $A = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$; applying it to $(4/3, 1)^T$ gives $(5, 12)^T$. If $PA = LU$, multiply the original equation by P and solve $Ly = Pb$, followed by $Ux = y$. Omitting P on b solves a different system.

3.1 Rank B is two. Restrict B to $\ker(AB)$ to get

$$\dim \ker(AB) = \dim \ker B + \dim(\text{im } B \cap \ker A).$$

The two subspaces in the intersection both have dimension two in five dimensions, so intersection dimension may be zero, one or two. Thus nullity is 3, 4 or 5. To realise each, take $\ker A = \text{span}(e_1, e_2)$ and image B respectively $\text{span}(e_3, e_4)$, $\text{span}(e_1, e_3)$ or $\text{span}(e_1, e_2)$. Choose any rank-two B onto that image. These constructions prove attainability rather than only a loose interval bound.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.