

15.1 Taylor expansions and power-series endpoints

Name: _____ Class: _____ Date: _____

Total: 10 marks**KEY WORDS** radius of convergence • remainder

Objective

Construct Taylor polynomials and control approximation error; Determine radii of convergence and test endpoints separately; Differentiate and integrate power series within their interval of convergence.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

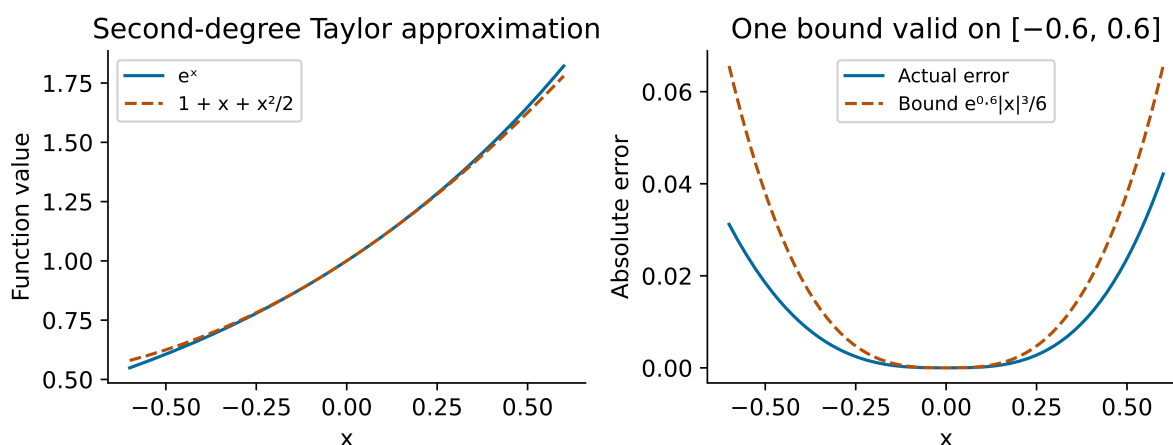
1 Key knowledge and worked examples

radius of convergence : The distance from the series centre inside which a power series converges absolutely.

remainder : The difference between a function and its finite approximation.

The Taylor polynomial of degree m at a is the sum of $f^{(k)}(a)(x-a)^k/k!$ for k from zero to m . The factorial belongs to each coefficient. If the next derivative is bounded in magnitude by M between a and x , the Lagrange remainder has magnitude at most $M|x-a|^{m+1}/(m+1)!$. Smoothness alone does not guarantee that the infinite Taylor series equals the function everywhere.

For $\sum x^n/n$ with $n \geq 1$, the ratio test gives radius 1. At $x=1$ it is the divergent harmonic series; at $x=-1$ it is an alternating convergent series. Thus its real convergence interval is $[-1,1)$. Differentiating inside gives $\sum x^{n-1}=1/(1-x)$, which converges at neither endpoint. The radius stayed 1 while the endpoint inclusion changed.



Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 What is the coefficient of x^3 in the Maclaurin series of $e^{(2x)}$? [1]

- A $1/6$
- B $1/3$
- C $2/3$
- D $4/3$
- E 8

2.2 Which is the interval of convergence of $\sum x^n/n^2$ for $n \geq 1$? [1]

- A $(-1,1)$
- B $[-1,1)$
- C $(-1,1]$
- D $[-1,1]$
- E $\mathbb{R}$

2.3 Find the convergence interval of $\sum_{n=1}^{\infty} x^n/n$ and of its termwise derivative. Approximate $e^{0.1}$ by its quadratic Taylor polynomial with a justified error bound. [4]

3 Exam-style questions

3.1 Evaluate $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$. Explain why replacing cosine by only $1 - x^2/2$ cannot determine this limit. [4]
