

Solutions

2.1 D. The quadratic x^2+x+1 has no root among 0 and 1. The others factor as $x \cdot x$, $(x+1)^2$, $x(x+1)$ and $x^2(x+1)$, respectively, over F_2 . The four rejected polynomials factor respectively as x times x , $(x+1)$ squared, $x(x+1)$, and x squared times $(x+1)$ in characteristic two.

2.2 E. The tower law multiplies finite degrees, giving 2 times 3=6. One, two and three omit a nontrivial extension factor; five adds degrees instead of multiplying them.

2.3 The quadratic takes value one at both elements of $\mathbb{F}_2$, so it has no root and is irreducible. Its quotient is a field. Division leaves a representative of degree below two: $0, 1, \alpha, 1 + \alpha$. The relation is $\alpha^2 = \alpha + 1$ in characteristic two. Consequently $\alpha(\alpha + 1) = \alpha^2 + \alpha = 1$, so $\alpha^{-1} = \alpha + 1$. This is not $\mathbb{Z}/4\mathbb{Z}$, which has a nonzero zero divisor.

3.1 Direct multiplication gives $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$, so it is reducible over $\mathbb{Q}$ despite having no real, hence no rational, roots. The no-root test characterises irreducibility only for degree two or three over a field. If an irreducible cubic root β belonged to a degree-two extension L , then $\mathbb{Q} \subset \mathbb{Q}(\beta) \subset L$ and the tower law would make 3 divide 2. This is impossible. The conclusion uses irreducibility, not merely a cubic equation.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.