

8.1 Differential equations and initial conditions

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS initial condition • characteristic root

Objective

Solve separable first-order equations; Solve constant-coefficient second-order equations; Use initial conditions and uniqueness conditions.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

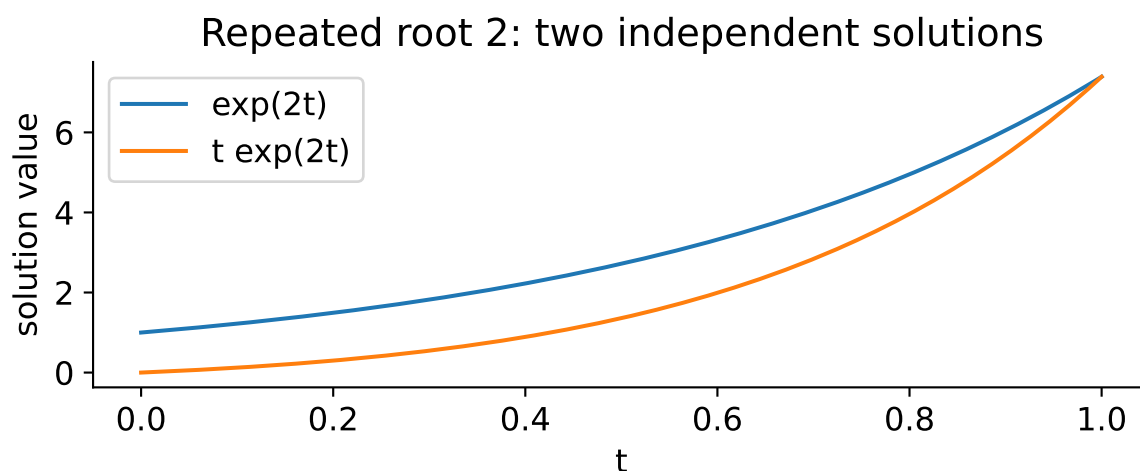
1 Key knowledge and worked examples

initial condition : A value of the solution or derivative specified at a starting point.

characteristic root : A root of the polynomial governing exponential solutions.

For $y' = ky$, separation and integration give $y = Ce^{(kt)}$; recover the zero solution if division by y was used. An initial value determines C .

Solve $y'' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 4$. The characteristic roots are $\pm 2i$, so $y = A \cos 2t + B \sin 2t$. The first condition gives $A = 3$; differentiating gives $y'(0) = 2B = 4$, so $B = 2$.



Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 For $y' = 2y$ and $y(0) = 3$, the solution is: [1]

- **A** $3e^{(2t)}$
 - **B** $2e^{(3t)}$
 - **C** $3 + 2t$
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2.2 For a repeated root $r = -1$, the general solution includes: [1]

- **A** $C_1 e^{(-t)} + C_2 e^{(-t)}$
 - **B** $(C_1 + C_2 t) e^{(-t)}$
 - **C** $C_1 \cos t + C_2 \sin t$
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2.3 Solve $y'' - 4y' + 4y = 0$ with $y(0) = 1$, $y'(0) = 0$. Check both initial conditions and the equation. [4]

3 Exam-style questions

3.1 Show that the initial-value problem $y' = 2\sqrt{|y|}$, $y(0) = 0$ has more than one solution for $t \geq 0$. Identify the missing standard uniqueness hypothesis. [4]

Solutions

2.1 A. The initial condition fixes $C=3$. The second exponential has wrong initial value and rate; the linear expression has derivative two but twice its value is not constantly two.

2.2 B. The factor t supplies a second independent solution. Two multiples of the same exponential combine into one solution. Sine and cosine satisfy the characteristic equation $r^2 + 1 = 0$, not a repeated root minus one.

2.3 The characteristic polynomial is $(r - 2)^2$. Independent solutions are e^{2t} and te^{2t} , hence $y = (C_1 + C_2t)e^{2t}$. The initial value gives $C_1 = 1$; the derivative at zero gives $2C_1 + C_2 = 0$, so $C_2 = -2$.

$$y(t) = (1 - 2t)e^{2t}, \quad y'(t) = -4te^{2t}, \quad y''(t) = (-4 - 8t)e^{2t}.$$

Substitution gives $y'' - 4y' + 4y = 0$, and the two initial values hold. Two copies of e^{2t} would not span the solution space.

3.1 One solution is identically zero. For any $a \geq 0$, set $y_a(t) = 0$ for $0 \leq t \leq a$ and $y_a(t) = (t - a)^2$ for $t > a$. The derivative is zero on the first interval and $2(t - a)$ on the second; both sides have derivative zero at a . Therefore y_a is differentiable and satisfies the equation and initial value. The right side is continuous but is not locally Lipschitz in y at zero: $|2\sqrt{h} - 0|/|h| = 2/\sqrt{h}$ is unbounded. Existence does not imply uniqueness, and division by $\sqrt{y}$ would lose the waiting-time solutions.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.