

19.1 Functions, inverse branches and composition

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS bijection • involution

Objective

Distinguish injectivity and surjectivity using the stated domain and codomain; Choose and verify an inverse branch by composing in both directions; Trace repeated function composition while preserving the domain.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

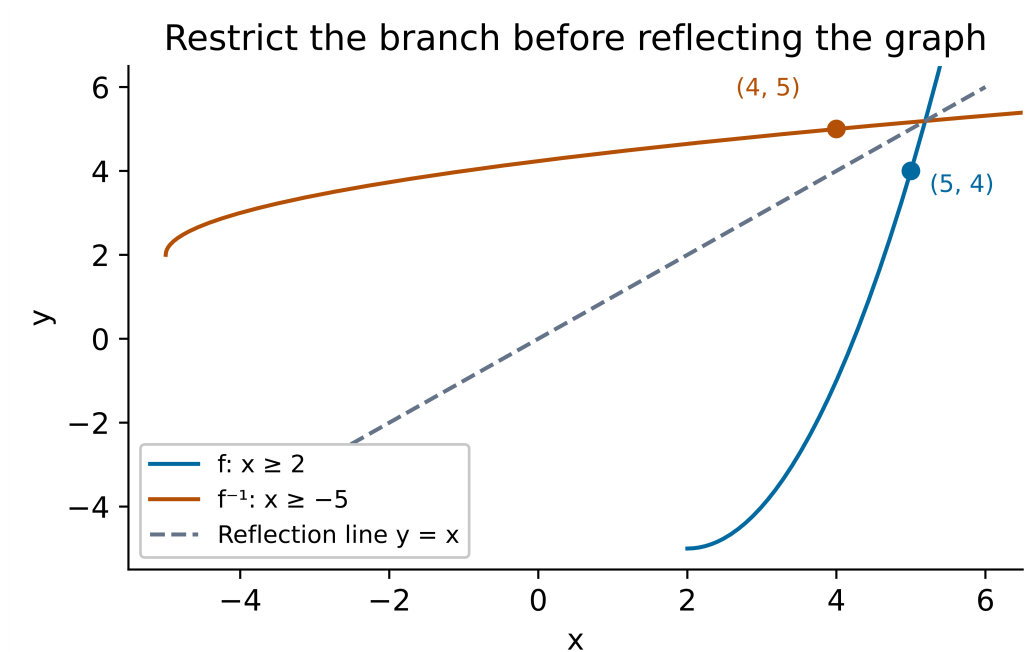
1 Key knowledge and worked examples

bijection : A function that is both injective and surjective between its specified sets.

involution : A function whose composition with itself is the identity on its domain.

A function assigns exactly one output to each allowed input. Injective means equal outputs force equal inputs; surjective means every element of the stated codomain is reached. The map $x \mapsto x^2$ from $\mathbb{R}$ to $[0, \infty)$ is surjective but not injective. From $[0, \infty)$ to $[0, \infty)$ it is both, hence bijective. From $[0, \infty)$ to $\mathbb{R}$ it remains injective but fails surjectivity. Keep domain, image and codomain separate when testing each claim.

For $f(x) = (x-2)^2 - 5$ with $x \geq 2$, $f(5) = 4$ and $f^{-1}(4) = 2 + \sqrt{9} = 5$. The unrestricted quadratic would have two inputs, -1 and 5 , giving output 4 . For $g(x) = 1/(1-x)$, the orbit $3 \rightarrow -1/2 \rightarrow 2/3 \rightarrow 3$ has period three, so $g^8(3) = 2/3$; the count concerns compositions, not eighth powers.



Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 For $f: [2, \infty) \rightarrow [-5, \infty)$, $f(x) = (x-2)^2 - 5$, what is $f^{-1}(y)$? [1]

- **A** $2 + \sqrt{(y+5)}$
- **B** $2 - \sqrt{(y+5)}$
- **C** $\sqrt{(y-5)}$
- **D** $1/((y-2)^2 - 5)$
- **E** $(y+2)^2 - 5$

2.2 For $g(x) = 1/(1-x)$ on $\mathbb{R}$ excluding 0 and 1, what is $g^8(3)$? [1]

- **A** 3
- **B** $-1/2$
- **C** $2/3$
- **D** $1/256$
- **E** 9

2.3 Restrict $f(x) = (x-1)^2 + 2$ to $x \geq 1$. State its image and inverse, including domains. Verify both inverse compositions. [4]

3 Exam-style questions

3.1 Let $h(x) = 1/(1-x)$. Determine the domain of $h \circ h \circ h$ and its formula. May the simplified expression be extended to the excluded points without changing the original composite? [4]

Solutions

2.1 A. The input restriction $x \geq 2$ forces $x - 2 \geq 0$, so select the positive square-root branch. The negative branch violates $x \geq 2$; square root of $y - 5$ has wrong shift. A reciprocal is not an inverse function, and the last quadratic does not undo the original square.

2.2 C. The orbit is 3, $-1/2$, $2/3$, then 3 again. Eight is congruent to two modulo three. Three is the zero-mod-three iterate; minus one-half is the one-mod-three iterate. One over 256 treats composition as multiplication, and nine squares the input instead of iterating the map.

2.3 The image is $[2, \infty)$. Solving $y = (x - 1)^2 + 2$ with $x \geq 1$ selects $x = 1 + \sqrt{y - 2}$. Thus $f^{-1} : [2, \infty) \rightarrow [1, \infty)$ has that formula. For allowed x , $1 + \sqrt{(x - 1)^2} = x$ because $x - 1 \geq 0$. For allowed y , $(\sqrt{y - 2})^2 + 2 = y$. The negative square-root branch belongs to a different restriction, not a second inverse on this domain.

3.1 The first stage requires $x \neq 1$. The second requires $h(x) \neq 1$, which excludes $x = 0$. For $x \neq 0, 1$, $h^2(x) = (x - 1)/x$. This never equals one, so the third stage adds no exclusion. Then $h^3(x) = x$ on $\mathbb{R} \setminus \{0, 1\}$. The identity formula is defined more widely, but adding 0 or 1 defines an extension, not the original composite. Each intermediate input must be permitted before cancellation.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.