

# 15.1 Taylor expansions and power-series endpoints

Name: \_\_\_\_\_ Class: \_\_\_\_\_ Date: \_\_\_\_\_

Total: 10 marks

KEY WORDS radius of convergence • remainder

## Objective

Construct Taylor polynomials and control approximation error; Determine radii of convergence and test endpoints separately; Differentiate and integrate power series within their interval of convergence.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

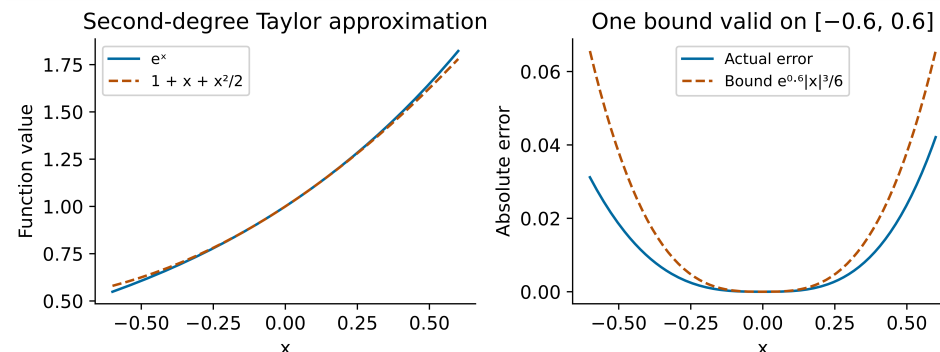
## 1 Key knowledge and worked examples

**radius of convergence** : The distance from the series centre inside which a power series converges absolutely.

**remainder** : The difference between a function and its finite approximation.

The Taylor polynomial of degree  $m$  at  $a$  is the sum of  $f^{(k)}(a)(x-a)^k/k!$  for  $k$  from zero to  $m$ . The factorial belongs to each coefficient. If the next derivative is bounded in magnitude by  $M$  between  $a$  and  $x$ , the Lagrange remainder has magnitude at most  $M|x-a|^{m+1}/(m+1)!$ . Smoothness alone does not guarantee that the infinite Taylor series equals the function everywhere.

For  $\sum x^n/n$  with  $n \geq 1$ , the ratio test gives radius 1. At  $x=1$  it is the divergent harmonic series; at  $x=-1$  it is an alternating convergent series. Thus its real convergence interval is  $[-1,1)$ . Differentiating inside gives  $\sum x^{n-1}=1/(1-x)$ , which converges at neither endpoint. The radius stayed 1 while the endpoint inclusion changed.



Original course illustration; its values belong to the worked example, not the later practice.

## 2 Practice

2.1 What is the coefficient of  $x^3$  in the Maclaurin series of  $e^{2x}$ ? [1]

- A  $1/6$
- B  $1/3$
- C  $2/3$
- D  $4/3$
- E 8

2.2 Which is the interval of convergence of  $\sum x^n/n^2$  for  $n \geq 1$ ? [1]

- A  $(-1,1)$
- B  $[-1,1)$
- C  $(-1,1]$
- D  $[-1,1]$
- E  $\mathbb{R}$

**2.3** Find the convergence interval of  $\sum_{n=1}^{\infty} x^n/n$  and of its termwise derivative. Approximate  $e^{0.1}$  by its quadratic Taylor polynomial with a justified error bound. [4]

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### 3 Exam-style questions

**3.1** Evaluate  $\lim_{x \rightarrow 0} (\cos x - 1 + x^2/2)/x^4$ . Explain why replacing cosine by only  $1 - x^2/2$  cannot determine this limit. [4]

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## Solutions

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**2.1 D.** The coefficient is  $2^3/3!=8/6=4/3$ ; the factorial cannot be omitted. One-sixth is the coefficient for  $e^x$ ; one-third and two-thirds omit some powers of two; eight omits division by factorial three.

**2.2 D.** The radius is 1 and both endpoints converge absolutely by comparison with sum  $1/n^2$ . Each open or half-open interval wrongly discards at least one absolutely convergent endpoint; all of  $\mathbb{R}$  includes inputs with magnitude greater than one where terms fail to tend to zero.

**2.3** The ratio test gives radius one. At 1 the original harmonic series diverges; at -1 the alternating series converges. Thus the original interval is  $[-1, 1)$ . The derivative is  $\sum_{n=1}^{\infty} x^{n-1} = 1/(1-x)$  only for  $|x| < 1$ ; its terms do not tend to zero at either endpoint.  $P_2(0.1) = 1+0.1+0.1^2/2 = 1.105$ . On  $[0, 0.1]$ , the third derivative is  $e^x < 2$ . The Lagrange bound gives  $|R_2| \leq 2(0.1)^3/6 = 1/3000$ . The bound is deliberately conservative, not a claim of exact error.

**3.1** Taylor's theorem near zero gives  $\cos x = 1 - x^2/2 + x^4/24 + O(x^6)$ . Subtraction cancels the constant and quadratic terms, leaving  $x^4/24 + O(x^6)$ . After division the limit is  $1/24$ . A truncation with remainder merely  $o(x^2)$  gives no control after division by  $x^4$ ; retain and bound the first surviving order.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.