

14.1 Open sets, compactness and connectedness

Name: _____ Class: _____ Date: _____

Total: 10 marks

KEY WORDS relative topology • connectedness

Objective

Compute closure, interior and boundary in a stated space; Apply compactness and connectedness to continuous maps; Distinguish relative topology from the ambient Euclidean topology.

Original public practice. Marking guidance is authored locally. Use the topic reference for prerequisites and full theorem statements.

1 Key knowledge and worked examples

relative topology : Open subsets inherited by intersecting a subspace with ambient open sets.

connectedness : Absence of a separation into disjoint nonempty relatively open subsets.

In a metric space, an open set contains a small ball around each of its points. The interior consists of such points; the closure includes all limit points; the boundary is closure minus interior. Open and closed are not mutually exclusive labels; the empty set and the whole space are both. In $\mathbb{R}$, the set $[0,1]$ has interior $(0,1)$, closure $[0,1]$ and boundary $\{0,1\}$.

Let $X=[0,1]$ with its usual relative topology. The set $U=[0,0.5)$ equals X intersected with $(-1,0.5)$, so U is open in X . Its closure in X is $[0,0.5]$, and its boundary in X is $\{0.5\}$; zero is an interior point relative to X . For a continuous f on X with $f(0)<0<f(1)$, connectedness ensures a zero, while compactness separately ensures attained extrema.

The ambient space changes the boundary



$$X = [0, 1]; U = [0, 1/2]$$

Relative interior: U Closure in X : $[0, 1/2]$

Boundary in X : $\{1/2\}$ Boundary in $\mathbb{R}$: $\{0, 1/2\}$

Original course illustration; its values belong to the worked example, not the later practice.

2 Practice

2.1 What is the boundary of $(0,1)$ as a subset of $\mathbb{R}$? [1]

- **A** The empty set
- **B** $\{0\}$
- **C** $\{1\}$
- **D** $\{0,1\}$
- **E** $[0,1]$

2.2 Which subset of $\mathbb{R}$ is connected but not compact? [1]

- **A** $\{0,1\}$
- **B** $[0,1]$
- **C** $(0,1)$
- **D** The empty set
- **E** $[0,1]$ union $[2,3]$

2.3 In $X = [0, 2]$ with its relative topology, find the interior, closure and boundary of $A = [0, 1)$. Is A open or closed in X ? [4]

3 Exam-style questions

3.1 Give a compact disconnected real set and a connected noncompact real set. Must a continuous real function on the latter attain an extreme value? Explain which property gives the intermediate value theorem. [4]

Solutions

2.1 D. The closure is $[0,1]$ and the interior is $(0,1)$; removing the latter leaves both endpoints. Empty omits both endpoints; the singleton options omit one endpoint; $[0,1]$ incorrectly includes interior points.

2.2 C. An open interval is connected but not closed in $\mathbb{R}$, so it is not compact. The other options are compact or disconnected. The finite two-point set and the disjoint closed intervals are disconnected; the closed interval and the empty set are compact. The empty set is connected under the usual convention but still fails noncompactness.

2.3 $A = X \cap (-1, 1)$ is open in X , so its relative interior is A . Its closure is $[0, 1]$ and its boundary is $\{1\}$. Zero is interior relative to X because a sufficiently small ball in X has no negative part. A is not closed in X : the sequence $1 - 1/n$ for $n \geq 2$ tends to the missing point 1. In the ambient real line, interior and boundary would instead be $(0, 1)$ and $\{0, 1\}$.

3.1 The finite set $\{0, 2\}$ is compact but disconnected. The interval $(0, 1)$ is connected but not compact. The continuous identity function on $(0, 1)$ attains neither a maximum nor a minimum. Connectedness makes a continuous real image an interval and therefore supplies intermediate values. Compactness instead makes a nonempty continuous real image attain its extreme values. Neither property substitutes for the other.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.