

Exercise walk-through

31.1

Metric completeness and closure from a basis

GRE · GRE Subject Test · GRE Mathematics

7. Choose and justify a method

Choose and justify a method

The pullback metric is complete exactly when the image $h(X)$ is complete in the ordinary real distance. A closed subset of $\mathbb{R}$ is complete. For $h(x)=\arctan x$ on $\mathbb{R}$, the image is $(-\pi/2, \pi/2)$, which omits its endpoints. The sequence n is Cauchy in the pullback metric because $\arctan n \rightarrow \pi/2$, but no real x has $\arctan x = \pi/2$, so the metric is incomplete. For $h(x)=x^3$, the image is all $\mathbb{R}$ and the pullback metric is complete. Topological equivalence alone does not preserve completeness.

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In a topology with a basis, x lies in the closure of A when every basic open neighbourhood of x meets A . This is a membership test, not automatically a Euclidean endpoint operation. The basis must cover the space, and for a point in two basis sets there must be a smaller basis set containing it within their intersection. A point can lie in the closure of a singleton even when it is not the singleton's own point; that possibility depends on the topology.

14. Reasoning and check

Reasoning and check

Both defining maps are injective, giving positive separation; absolute distance gives symmetry and the triangle inequality. Each map is an isometry onto its real image. The cube map has image all of $\mathbb{R}$, a complete space, so d_1 is complete. The arctangent image is the open interval $(-\pi/2, \pi/2)$, which is incomplete. The sequence n is d_2 -Cauchy because $\arctan n$ tends to $\pi/2$, but no real point has that image, so it has no d_2 -limit. Bounded metric values do not imply completeness.

Exercise 2.1

2.1 Which statement holds for $d(x,y)=|\arctan x-\arctan y|$ on $\mathbb{R}$?

[1]

- **A** It is not a metric
- **B** It is complete because $\arctan$ is continuous
- **C** It is incomplete because its isometric image omits its endpoints
- **D** Every unbounded sequence fails to be Cauchy
- **E** Its distance equals ordinary absolute difference

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2.1 C. The injective map defines a metric. The image is an open bounded interval, and $\arctan n$ tends to a missing endpoint. Injectivity proves this is a metric. Continuity does not give completeness, unbounded n is Cauchy in this metric, and arctangent distance is not ordinary absolute distance.

Exercises 2.2·2.3

2.2 For the divisor basis in the lesson, which point belongs to closure of $\{8\}$? [1]

- A 2
- B 3
- C 4
- D 6
- E 16

2.3 On $\mathbb{R}$ compare $d_1(x, y) = |x^3 - y^3|$ and $d_2(x, y) = |\arctan x - \arctan y|$. Prove they

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2.2 E. Closure consists of multiples of 8. The point 16 has every neighbourhood containing 8; U_4 demonstrates why the divisor 4 does not belong. Closure requires multiples of eight. Two, three, four and six have divisor neighbourhoods missing eight; sixteen is a multiple and every neighbourhood contains eight.

2.3 On $\mathbb{R}$ compare $d_1(x, y) = |x^3 - y^3|$ and $d_2(x, y) = |\arctan x - \arctan y|$. Prove they

2.3 Both defining maps are injective, giving positive separation; absolute distance gives symmetry and the triangle inequality. Each map is an isometry onto its real image. The cube map has image all of $\mathbb{R}$, a complete space, so d_1 is complete. The arctangent image is the open interval $(-\pi/2, \pi/2)$, which is incomplete. The sequence n is d_2 -Cauchy because $\arctan n$ tends to $\pi/2$, but no real point has that image, so it has no d_2 -limit. Bounded metric values do not imply completeness.

Exercise 3.1

3.1 On $X = \{2, 3, 4, \dots\}$ use the divisor basis $U_k = \{n \in X : n \mid k\}$. Find the closure of $\{6\}$ and the interior of $\{6\}$. Verify the basis conditions, including empty intersections.^[4]

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3.1 Every x belongs to U_x . An intersection is $U_{\gcd(k,l)}$ if the gcd is at least two, and is empty for gcd one. Thus a point in an intersection has a basic neighbourhood within it. The smallest basic neighbourhood of x is U_x , contained in every U_k that contains x . It meets $\{6\}$ exactly when $6 \mid x$. Therefore the closure is $\{6, 12, 18, \dots\}$, the multiples of six. The point six is not interior to its singleton because U_6 also contains two and three. No other point is in that singleton, so its interior is empty. This closure calculation uses all basic neighbourhoods, not Euclidean intuition.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.