

Exercise walk-through

30.1

Related rates and removable quotient limits

GRE · GRE Subject Test · GRE Mathematics

9. Choose and justify a method

Choose and justify a method

For $f(x)=x|x|$ and $g(x)=x$, both are continuously differentiable, $f(0)=g(0)=0$ and $g'(0)=1$. Their quotient for $x \neq 0$ is $|x|$, which extends continuously at zero but has unequal one-sided derivatives there. For $(f^2-f)/(2g-g^3)$, factor to $(f/g)(f-1)/(2-g^2)$; its removable limit is $-f'(0)/(2g'(0))$. Additional factors approach -1 and 2 . Distinguish the limit of the quotient from the derivative of its extension.

10. Worked reasoning

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For a sphere with $R=3$ and depth $h=1$, cross-sectional area is $\pi(6-1)=5\pi$. A depth decrease of $1/4$ per time unit therefore removes volume at magnitude $5\pi/4$ per time unit. For $f(x)=\sin(2x)$, $g(x)=3x$, the quotient extends at zero with value $2/3$. This limit says nothing about the size of a tank and should not be substituted as a derivative rate without identifying the dependent quantities.

16. Reasoning and check

Reasoning and check

Since $g'(0)$ is nonzero, the denominator is nonzero near zero except at zero. Factor the expression as $(f/g)(f-1)/(2-g^2)$. Its limit is $(4/2)(-1)/2 = -1$. First-order differentiability gives $f/g \rightarrow f'(0)/g'(0)$; it does not give second-order control. For a counterexample preserving these derivatives, take $f(x) = 4x + x|x|$ and $g(x) = 2x$. They are continuously differentiable. For nonzero x the quotient becomes $(2 + |x|/2)(-1 + 4x + x|x|)/(2 - 4x^2)$. Its expansion is $-1 + 4x - |x|/4 + O(x^2)$, giving different one-sided derivatives.

Exercise 2.1

2.1 A radius-3 spherical tank has depth 1 decreasing at $1/4$ per time unit. What is the positive outflow magnitude? [1]

- **A** $\pi/4$
- **B** $3\pi/4$
- **C** $5\pi/4$
- **D** $9\pi/4$
- **E** 36π

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2.1 C. Cross-section area is $\pi(2Rh-h^2)=5\pi$. Multiply by depth-decrease magnitude $1/4$. The section area is five pi and the speed magnitude one-quarter. The other coefficients use radius, whole-sphere quantities or omit the correct section factor.

Exercise 2.2

2.2 For $x \neq 0$, $f(x) = x|x|$, $g(x) = x$. What happens when f/g is filled in at zero? [1]

- **A** It stays undefined
- **B** It becomes discontinuous
- **C** It is continuous but not differentiable
- **D** It is differentiable with derivative 1
- **E** It is constantly zero

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- **B** It becomes discontinuous
- **C** It is continuous but not differentiable
- **D** It is differentiable with derivative 1
- **E** It is constantly zero

2.2 C. The quotient is $|x|$. Filling in zero gives continuity, while the left and right derivatives are -1 and 1 . Filling zero removes the missing value, so the extension is continuous. Its unequal one-sided slopes rule out differentiability with derivative one; it is not constant.

Exercise 2.3

2.3 A spherical tank has radius 2 m and water depth h measured from its bottom. Derive $V(h)$. At $h=1$ m and $dh/dt = -0.1$ m/min, find signed volume change and outflow magnitude. [4]

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2.3 At depth s , the cross-sectional radius obeys $r(s)^2 = 2Rs - s^2$. Integrating disks gives $V(h) = \pi(Rh^2 - h^3/3)$ for $0 \leq h \leq 2R$.

$$\frac{dV}{dt} = \pi(2Rh - h^2) \frac{dh}{dt}.$$

$$\frac{dV}{dt} = \pi[2(2 \text{ m})(1 \text{ m}) - (1 \text{ m})^2](-0.1 \text{ m/min}) = -0.3\pi \text{ m}^3/\text{min}.$$

The outflow magnitude is $0.3\pi \text{ m}^3/\text{min}$. The sign describes loss of stored water, not negative physical outflow.

Exercise 3.1

3.1 Suppose f, g are continuously differentiable near zero, $f(0) = g(0) = 0$, $f'(0) = 4$, $g'(0) = 2$. Find the removable limit of $(f^2 - f)/(2g - g^3)$. Do these assumptions imply differentiability of the extended quotient? Give a counterexample to that general implication. [4]

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Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.