

Exercise walk-through

3.1

Linear algebra

GRE · GRE Subject Test · GRE Mathematics

7. Choose and justify a method

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Row reduction exposes pivots and free variables without changing the solution set of a linear system. Rank is the dimension of the column image, equivalently the number of pivots. Nullity is the kernel dimension: for a map from an n -dimensional domain, $\text{rank} + \text{nullity} = n$. In a nonhomogeneous system, a zero coefficient row with nonzero right side means inconsistency; free variables give infinitely many solutions only after consistency has been established. The codomain dimension need not equal rank.

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An eigenvector is nonzero and satisfies $Av = \lambda v$, so eigenvalues are roots of $\det(A - \lambda I)$. Diagonalisation requires a full independent eigenvector basis over the chosen field. Distinct eigenvalues give independent eigenvectors, while repeated eigenvalues may have too small an eigenspace. A real symmetric matrix has a real orthonormal eigenbasis. A real odd-dimensional matrix has at least one real eigenvalue because its real characteristic polynomial has odd degree; that alone does not imply diagonalisation or all eigenvalues real.

16. Reasoning and check

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The characteristic polynomial is $(2 - \lambda)^2$ and $\det A = 4$. Solving $(A - 2I)v = 0$ gives $v_2 = 0$, so the eigenspace has dimension one over either field. Two independent eigenvectors are required; the matrix is invertible but not diagonalizable over either field. Write $A = 2I + N$ where $N^2 = 0$. The commuting binomial expansion leaves only two terms: $A^n = 2^n I + n2^{n-1}N = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}$. Multiplying this expression by A gives the formula at $n + 1$, independently checking the coefficient.

Exercise 2.1

2.1 A 3-column matrix has rank 2. Its nullity is:

[1]

- A 1
- B 2
- C 3
- D 0
- E 5

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2.1 A. Rank-nullity gives $3-2=1$. B, C and E exceed the one free column allowed by rank two; D would require three independent columns.

Exercises 2.2·2.3

2.2 $\det A=0$ implies:

[1]

- **A** A is invertible
- **B** A is singular
- **C** All entries are zero
- **D** Every column is zero
- **E** Every eigenvalue is nonzero

2.3 For $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $T(x, y, z) = (x + 2y + z, 2x + 4y + 2z)$, find bases of the kernel

Exercises 2.2-2.3

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2.2 B. A zero determinant means the square matrix is singular. It can have many nonzero entries; a single dependence is enough. A and E contradict singularity. C and D are stronger than dependence and fail, for example, for $\text{diag}(1,0)$.

2.3 For $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with $T(x, y, z) = (x + 2y + z, 2x + 4y + 2z)$, find bases of the kernel

2.3 The second row is twice the first, so rank is one. The homogeneous equation gives $x = -2y - z$.

$$\ker T = \text{span}\{(-2, 1, 0), (-1, 0, 1)\}.$$

The two displayed vectors are independent. Every output is $(s, 2s)$, and $(1, 2)$ occurs; this is an image basis. Rank plus nullity is $1 + 2 = 3$, the domain dimension. For $(3, 6)$, all solutions are $(3 - 2a - b, a, b)$, with $a, b \in \mathbb{R}$. For $(3, 7)$ there is no solution, since $7 \neq 2 \cdot 3$.

Exercise 3.1

3.1 Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Determine its eigenvalues, determinant and diagonalisation over $\mathbb{R}$ and $\mathbb{C}$. Find A^n for integers $n \geq 1$ without repeated multiplication. [4]

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3.1 The characteristic polynomial is $(2 - \lambda)^2$ and $\det A = 4$. Solving $(A - 2I)v = 0$ gives $v_2 = 0$, so the eigenspace has dimension one over either field. Two independent eigenvectors are required; the matrix is invertible but not diagonalizable over either field. Write $A = 2I + N$ where $N^2 = 0$. The commuting binomial expansion leaves only two terms:

$$A^n = 2^n I + n2^{n-1}N = \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix}.$$

Multiplying this expression by A gives the formula at $n + 1$, independently checking the coefficient.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.