

Exercise walk-through

# 29.1

## Integration by parts, order reversal and symmetry

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GRE · GRE Subject Test · GRE Mathematics

## 6. Choose and justify a method

### Choose and justify a method

Integration by parts comes from the product rule:  $\int u \, dv = uv - \int v \, du$ . For definite integrals include the boundary term at both ends. With  $I_n(x) = \int_1^x (\ln t)^n \, dt$ , choose  $u = (\ln t)^n$  and  $dv = dt$ . For  $n \geq 1$  the lower boundary vanishes, giving  $I_n = x(\ln x)^{n-1} - I_{n-1}$ . A plus sign would violate the differentiated identity. Keep the same lower limit in a recurrence; changing it changes the constants.

## 8. Choose and justify a method

### Choose and justify a method

To reverse  $\int_0^1 \int_x^1 F(x,y) dy dx$ , describe the triangle  $0 \leq x \leq y \leq 1$ , then rewrite it as  $\int_0^1$  integral from 0 to  $yF(x,y) dx dy$ . Both the outer interval and inner bounds change; swapping symbols alone changes the region. For  $F=e^{y^2}$ , integrating over  $x$  first gives  $\int_0^1 y e^{y^2} dy = (e-1)/2$ . Continuity on this compact triangle makes order reversal valid; singular or conditionally convergent cases require stronger care.

## 14. Reasoning and check

### Reasoning and check

The region is  $0 \leq x \leq y \leq 1$ . In the reversed order,  $y$  ranges from zero to one and  $x$  from zero to  $y$ . Continuity on the compact triangle permits order reversal, giving  $\int_0^1 ye^{y^2} dy = (e - 1)/2$ . Integration by parts with  $u = (\ln t)^2$ ,  $dv = dt$  gives  $I_2 = x(\ln x)^2 - 2I_1$ . Since  $I_1 = x \ln x - x + 1$ ,  $I_2 = x[(\ln x)^2 - 2 \ln x + 2] - 2$ . Differentiation checks the integrand; substituting  $x=1$  checks the constant.

## Exercise 2.1

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**2.1** If  $I_n(x) = \int_1^x (\ln t)^n dt$  for  $n \geq 1$ , which recurrence is correct?

[1]

- **A**  $x(\ln x)^{n+1} + nI_{n-1}(x)$
- **B**  $x(\ln x)^{n-1} - nI_{n-1}(x)$
- **C**  $(\ln x)^{n-1} - I_{n-1}(x)$
- **D**  $xI_{n-1}(x)$
- **E**  $nI_{n-1}(x)$

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**2.1** If  $I_n(x) = \int_1^x (\ln t)^n dt$  for  $n \geq 1$ , which recurrence is correct? [1]

- **A**  $x(\ln x)^{n+n}I_{n-1}$
- **B**  $x(\ln x)^{n-n}I_{n-1}$
- **C**  $(\ln x)^{n-I_{n-1}}$
- **D**  $xI_{n-1}$
- **E**  $nI_{n-1}$

**2.1 B.** Integration by parts gives  $uv$  minus the remaining integral. The lower boundary is zero for  $n \geq 1$ . The plus-sign choice violates integration by parts; the version without  $x$  has wrong boundary term. Multiplying by  $x$  or by  $n$  alone omits the required boundary expression.

## Exercises 2.2·2.3

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**2.2** After reversing  $\int_0^1 \int_x^1 e^{\widehat{(y^2)}} dy \, dx$ , what integral remains?

[1]

- **A**  $\int_0^1 e^{\widehat{(y^2)}} dy$
- **B**  $\int_0^1 y \, e^{\widehat{(y^2)}} dy$
- **C**  $\int_0^1 (1-y) e^{\widehat{(y^2)}} dy$
- **D**  $\int_0^1 y^2 \, e^{\widehat{(y^2)}} dy$
- **E**  $\int_0^1 e^{\widehat{y}} dy$

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**2.3** Evaluate  $\int_0^1 \int_x^1 e^{y^2} dy \, dx$  by reversing order. Describe the region before and after

## Exercises 2.2·2.3

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**2.2** After reversing  $\int_0^1 \int_x^1 e^{-(y^2)} dy dx$ , what integral remains? [1]

- **A**  $\int_0^1 e^{-(y^2)} dy$
- **B**  $\int_0^1 y e^{-(y^2)} dy$
- **C**  $\int_0^1 (1-y) e^{-(y^2)} dy$
- **D**  $\int_0^1 y^2 e^{-(y^2)} dy$
- **E**  $\int_0^1 e^{-y} dy$

**2.2 B.** At each  $y$ ,  $x$  ranges from 0 to  $y$ . The inner integral supplies length  $y$ . Reversal gives inner length  $y$ . Unit length, one minus  $y$  and  $y$  squared describe different regions;  $e$  to  $y$  changes the given integrand.

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**2.3** The region is  $0 \leq x \leq y \leq 1$ . In the reversed order,  $y$  ranges from zero to one and  $x$  from zero to  $y$ . Continuity on the compact triangle permits order reversal, giving  $\int_0^1 y e^{y^2} dy = (e - 1)/2$ . Integration by parts with  $u = (\ln t)^2$ ,  $dv = dt$  gives  $I_2 = x(\ln x)^2 - 2I_1$ . Since  $I_1 = x \ln x - x + 1$ ,

$$I_2 = x[(\ln x)^2 - 2 \ln x + 2] - 2.$$

Differentiation checks the integrand; substituting  $x=1$  checks the constant.

## Exercise 3.1

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**3.1** For positive integer  $m$ , evaluate  $\int_0^\pi \sin(2mx)/\sin x \, dx$ . Justify endpoint integrability

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**3.1** The endpoint limits are  $2m$  at zero and  $-2m$  at  $\pi$ , by the first-order sine limit. Both singularities are removable, so the continuously extended integrand is integrable. Reflection  $x \mapsto \pi - x$  preserves the denominator and reverses the numerator. The integral equals its negative and therefore is zero. Symmetric cancellation would not establish convergence for a genuinely divergent endpoint; the removable limits are essential.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.