

Exercise walk-through

28.1

LU factorisation and nullity of composed maps

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7. Choose and justify a method

Choose and justify a method

In $Ly=b$, compute y from top to bottom, subtracting terms already known; divide by the current diagonal unless it is one. In $Ux=y$, compute x from bottom to top. For each equation substitute back into the original row as a check. Reusing LU for many right sides avoids repeating the elimination: dense factorisation takes cubic-order work in dimension, while each triangular solve takes quadratic-order work. Exact arithmetic can check small examples; numerical pivoting reduces some roundoff problems but cannot remove inherent ill-conditioning.

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For endomorphisms of $\mathbb{R}^6$ with nullity $A=2$ and nullity $B=3$, rank $B=3$ and the intersection dimension can range from 0 to 2. Thus nullity of $A \circ B$ ranges from 3 to 5. Bounds depend on the common intermediate space: for subspaces of dimensions r and s in an m -dimensional space, their intersection has dimension between $\max(0, r+s-m)$ and $\min(r, s)$. Reversing the composition can change its nullity, even though AB and BA are both defined. Choose compatible spaces before applying these formulas.

16. Reasoning and check

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Rank B is two. Restrict B to $\ker(AB)$ to get $\dim \ker(AB) = \dim \ker B + \dim(\operatorname{im} B \cap \ker A)$. The two subspaces in the intersection both have dimension two in five dimensions, so intersection dimension may be zero, one or two. Thus nullity is 3, 4 or 5. To realise each, take $\ker A = \operatorname{span}(e_1, e_2)$ and image B respectively $\operatorname{span}(e_3, e_4)$, $\operatorname{span}(e_1, e_3)$ or $\operatorname{span}(e_1, e_2)$. Choose any rank-two B onto that image.

Exercise 2.1

2.1 For the stated unit-diagonal L and $b=(-1,-1,12)$, what is the second coordinate of y solving $Ly=b$? [1]

- **A** -3
- **B** -1
- **C** 0
- **D** 1
- **E** 12

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2.1 D. $y_1=-1$ and the second row is $2y_1+y_2=-1$. Therefore $y_2=1$. Substituting $y_1=-1$ in $2y_1+y_2=-1$ gives $y_2=1$. Minus three, minus one, zero and twelve fail that second row.

Exercise 2.2

2.2 A, B are maps $\mathbb{R}^6 \rightarrow \mathbb{R}^6$ with nullity A=2 and nullity B=3. What range is possible for nullity of $A \circ B$? [1]

- **A** 0 to 2
- **B** 2 to 3
- **C** 3 to 5
- **D** 5 to 6
- **E** Always 5

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2.2 C. Nullity B contributes 3, and $\text{im } B \cap \ker A$ has dimension from 0 to 2 in this ambient space. Nullity cannot be below three because $\ker B$ is included. It cannot exceed five because at most two new kernel dimensions arise from $\ker A$. Always five ignores images disjoint from $\ker A$.

Exercise 2.3

2.3 Let $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $b = (5, 12)^T$. Solve $LUx=b$ and verify in the multiplied matrix. For $PA=LU$, where does P act when solving $Ax=b$? [4]

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2.3 First solve $Ly=b$: $y_1 = 5$, $2y_1 + y_2 = 12$, so $y_2 = 2$. Then $Ux=y$ gives $x_2 = 1$ and $3x_1 + 1 = 5$, hence $x_1 = 4/3$. The product matrix is $A = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$; applying it to $(4/3, 1)^T$ gives $(5, 12)^T$. If $PA=LU$, multiply the original equation by P and solve $Ly=Pb$, followed by $Ux=y$. Omitting P on b solves a different system.

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$$\dim \ker(AB) = \dim \ker B + \dim(\operatorname{im} B \cap \ker A).$$

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Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.