

Exercise walk-through

27.1

Implicit differentiation and the inverse Jacobian

GRE · GRE Subject Test · GRE Mathematics

8. Choose and justify a method

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For implicit equations $H(u,v,x,y)=0$ and $K(u,v,x,y)=0$, differentiate while fixing the requested independent coordinate. Solve $[[H_u, H_v], [K_u, K_v]][u_x, v_x]^T = -[H_x, K_x]^T$ (transpose). The determinant in the unknown variables u, v must be nonzero to use the usual implicit-function theorem. Signs on the right come from moving known derivatives to the other side. If the determinant vanishes, this theorem is inconclusive; it does not by itself prove no inverse or no implicit solution exists.

9. Choose and justify a method

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The scalar shortcut $du/dx = 1/(dx/du)$ holds for a one-variable inverse with nonzero derivative, but usually fails for a coupled system because v changes to keep y fixed. For $x = u + v, y = u + 2v$, $J = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ has determinant 1 and inverse $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Therefore $u_x = 2$ while $1/f_u = 1$. An inverse Jacobian transforms differential sensitivities; a change-of-variables integral instead uses the absolute determinant for area or volume scaling, not a selected inverse entry.

14. Reasoning and check

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The corresponding output is $(1, 1)$. The Jacobian is $J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ with determinant -3, nonzero. Continuous partials give a differentiable local inverse. Thus

$J^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}$. Read rows as (u_x, u_y) and (v_x, v_y) , all at output $(1, 1)$.

Multiplication gives the identity. In particular $u_x = 1/3$, not $1/x_u = 1/2$: v changes to keep y fixed.

Exercise 2.1

2.1 For $x=u+v$, $y=u+2v$, what is u_x when y is held fixed?

[1]

- **A** 0
- **B** $1/2$
- **C** 1
- **D** 2
- **E** -1

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2.1 D. The inverse matrix of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Its top-left entry is $u_x=2$. Holding y fixed forces u and v to change together. The top-left inverse entry is two, ruling out zero, one-half, one and minus one; one-half would invert a single partial while fixing the wrong variable.

Exercises 2.2·2.3

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2.2 A. $\det J = -3$. The bottom-right inverse entry is $2/(-3) = -2/3$. The determinant is minus three and the required cofactor is two. The other choices use a different entry, miss the determinant or reverse its sign.

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Read rows as (u_x, u_y) and (v_x, v_y) , all at output $(1, 1)$. Multiplication gives the identity. In particular $u_x = 1/3$, not $1/x_u = 1/2$: v changes to keep y fixed.

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3.1 $\det DF = 4(u^2 + v^2) > 0$ away from zero, so the inverse-function theorem supplies a local inverse at each such point. But $F(u, v) = F(-u, -v)$, so it is not globally injective on the punctured plane. A vanishing determinant at zero means this theorem cannot guarantee a differentiable local inverse there. It is not, by itself, a general proof that no local set-theoretic inverse can exist for any map; for example the scalar cube map is bijective despite derivative zero at zero.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.