

Exercise walk-through

25.1

Integrating factors and nonhomogeneous differential equations

GRE · GRE Subject Test · GRE Mathematics

7. Choose and justify a method

Choose and justify a method

For $y' + 2xy = x$ with $y(0) = 1$, $\mu = e^{x^2}$. Then $(e^{x^2}y)' = x e^{x^2}$, whose integral is $e^{x^2}/2 + C$. Thus $y = 1/2 + C e^{-x^2}$, and the initial condition gives $C = 1/2$. Direct substitution checks $y' + 2xy = x$. The integrating factor makes the left side a product derivative; it is not an extra factor that remains multiplying the original right side in the final answer.

8. Choose and justify a method

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For $ay'' + by' + cy = r(x)$ with $a \neq 0$, solve the characteristic equation $a\lambda^2 + b\lambda + c = 0$ for the homogeneous part. Distinct real roots give two exponentials; a repeated root λ gives $(C_1 + C_2x)e^{\lambda x}$; roots $\alpha \pm i\beta$ give $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$. The complete solution is $y_h + y_p$. A polynomial forcing suggests a polynomial particular trial; an exponential or sine/cosine forcing suggests the corresponding family, with enough coefficients to account for differentiation.

9. Choose and justify a method

Choose and justify a method

If a particular trial duplicates a homogeneous solution, it cannot produce the forcing: multiply by x once for a simple root, twice for a repeated root. For $y'' - 3y' + 2y = e^x$, the naive Ke^x is annihilated. Trying Kxe^x gives $-Ke^x$, so $y_p = -xe^x$. Initial conditions determine the homogeneous constants only after adding y_p . Substitute the final function into the original equation to check signs and forcing; an initial-value check alone cannot verify the differential equation.

Exercises 2.1·2.2

2.1 Which integrating factor works for $y' + 3y = x$?

[1]

- A x^3
- B $e^{(3x)}$
- C $e^{(-3x)}$
- D $3x$
- E $1/x$

2.2 For $y'' - 3y' + 2y = 4$, which constant is a particular solution?

[1]

- A -4
- B -2
- C 0
- D 2
- E 4

Exercises 2.1·2.2

2.1 Which integrating factor works for $y' + 3y = x$? [1]

- **A** x^3
- **B** $e^{(3x)}$
- **C** $e^{(-3x)}$
- **D** $3x$
- **E** $1/x$

2.1 B. $p(x)=3$, so $\mu = \exp(\int 3 \, dx) = e^{(3x)}$. Its derivative is 3μ . A valid integrating factor

2.2 For $y'' - 3y' + 2y = 4$, which constant is a particular solution? [1]

- **A** -4
- **B** -2
- **C** 0
- **D** 2
- **E** 4

2.2 D. Both derivatives vanish for a constant. The equation reduces to $2y=4$, so $y=2$. For constants the derivatives vanish and $2y=4$. Minus four, minus two, zero and four fail this substitution.

Exercises 2.3·3.1

2.3 Solve $y' + 2y = e^x$, $y(0) = 1$, by an integrating factor. Check the answer directly. [4]

3 Exam-style questions

3.1 Solve $y'' - 2y' + y = e^x$ with $y(0) = 0$, $y'(0) = 0$. Explain why neither Ke^x nor Kxe^x is a sufficient particular trial. [4]

Exercises 2.3·3.1

2.3 Solve $y' + 2y = e^x$, $y(0) = 1$, by an integrating factor. Check the answer directly. [4]

3 Exam-style questions

2.3 The coefficients are continuous everywhere. The integrating factor is $\mu = e^{2x}$. Multiplication gives $(e^{2x}y)' = e^{3x}$. Integration gives $y = e^x/3 + Ce^{-2x}$. The initial condition yields $C=2/3$, so $y = e^x/3 + 2e^{-2x}/3$. Differentiation gives $y' = e^x/3 - 4e^{-2x}/3$; adding $2y$ gives e^x , and $y(0)=1$.

3.1 Solve $y'' - 2y' + y = e^x$ with $y(0) = 0$, $y'(0) = 0$. Explain why neither Ke^x nor Kxe^x is a sufficient particular trial. [4]

3.1 The homogeneous root is 1 with multiplicity two, so $y_h = (C_1 + C_2x)e^x$. Both suggested trials already belong to that homogeneous space and are annihilated. Put $y = e^xv$; then $y'' - 2y' + y = e^xv''$. Therefore $v'' = 1$ and $v = x^2/2 + C_2x + C_1$. The two initial conditions give $C_1 = C_2 = 0$, hence $y = x^2e^x/2$. This substitution also checks the forcing without relying only on initial values.

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.