

Exercise walk-through

20.1

Set images, equivalence relations and logical negation

GRE · GRE Subject Test · GRE Mathematics

7. Choose and justify a method

Choose and justify a method

Preimages behave differently. For $C \subseteq Y$, $f^{-1}(C)$ here denotes the set of all inputs sent into C , even if f has no inverse function. Membership in two preimages means the very same input maps into both target sets. Consequently preimages preserve unions, intersections and complements relative to the stated domain/codomain. Do not transfer a theorem about preimages to images. If f is injective, image intersections do become equal, because equal outputs then force a shared input.

8. Choose and justify a method

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A relation R on X is reflexive when xRx for every x , symmetric when xRy implies yRx , and transitive when xRy and yRz imply xRz . All three make an equivalence relation; its classes partition X . On integers, xRy when x and y have the same remainder modulo four gives four classes. The relation $|x-y| \leq 1$ is reflexive and symmetric but not transitive: $0R1$ and $1R2$ while 0 is not related to 2 . Checking only a diagram or two properties is insufficient.

9. Choose and justify a method

Choose and justify a method

An implication $P \Rightarrow Q$ is false exactly when P is true and Q false. Thus the negation of $P \Rightarrow (Q \wedge R)$ is $P \wedge (\neg Q \vee \neg R)$, not $\neg P \Rightarrow (\neg Q \wedge \neg R)$. Negating “every x has property A ” gives “there exists x without A ”; negating “there exists x ” gives “every x does not”. A counterexample can disprove a universal claim, while examples cannot prove it. The quantifiers retain their order when individually negated: $\neg(\forall x \exists y S(x,y))$ is $\exists x \forall y \neg S(x,y)$.

Exercise 2.1

2.1 Which identity holds for every function f and subsets A, B of its domain?

[1]

- **A** $f(A \cap B) = f(A) \cap f(B)$
- **B** $f(A \cup B) = f(A) \cup f(B)$
- **C** $f(A) = A$
- **D** $f(A) \cap f(B) = \emptyset$ whenever $A \cap B = \emptyset$
- **E** $f(A) \subseteq A$

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2.1 B. An output of an input in the union comes from at least one subset. Intersection equality can fail when distinct inputs share an output. A and D fail for a constant map on distinct inputs. C and E compare domain subsets with outputs without any identity-map or containment assumption.

Exercises 2.2·2.3

2.2 What is the negation of $P \Rightarrow (Q \wedge R)$?

[1]

- **A** $\neg P \wedge Q \wedge R$
- **B** $\neg P \Rightarrow (\neg Q \wedge \neg R)$
- **C** $P \wedge (\neg Q \vee \neg R)$
- **D** $P \vee \neg Q \vee \neg R$
- **E** $P \wedge \neg Q \wedge \neg R$

2.3 Prove $f(A \cap B) \subseteq f(A) \cap f(B)$. Give a strict-inclusion example. Under what additional assumption on f does equality follow for all A and B ?

[4]

Exercises 2.2-2.3

2.2 What is the negation of $P \Rightarrow (Q \wedge R)$? [1]

- **A** $\neg P \wedge Q \wedge R$
- **B** $\neg P \Rightarrow (\neg Q \wedge \neg R)$
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- **D** $P \vee \neg Q \vee \neg R$
- **E** $P \wedge \neg Q \wedge \neg R$

2.2 C. Failure of the implication requires a true premise and a false conjunction. De Morgan changes the latter into a disjunction. A makes the premise false; B is not the negation of the original implication; D allows a false premise. E requires both conclusions to fail, though failure of either is enough.

2.3 Prove $f(A \cap B) \subseteq f(A) \cap f(B)$. Give a strict-inclusion example. Under what additional assumption on f does equality follow for all A and B ? [4]

2.3 If $y = f(x)$ for $x \in A \cap B$, then x belongs to each set, so y belongs to each image. For strictness, map distinct a, b to the same c and take $A = \{a\}$, $B = \{b\}$. The left image is empty and the right intersection is $\{c\}$. If f is injective and $y = f(a) = f(b)$ with $a \in A, b \in B$, injectivity gives $a=b$ in the intersection, proving reverse inclusion. Conversely the singleton example shows that equality for all pairs forces injectivity.

Exercise 3.1

3.1 On the integers define aRb when $a - b$ is divisible by 4. Prove it is an equivalence relation and describe every class. Negate: “For every integer n there exists an integer m greater than n with property P .” [4]

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3.1 Reflexivity uses $a - a = 0$; symmetry uses the negative of a multiple of four; transitivity uses the sum of two such multiples. Classes are the four residues modulo four; the class of a is $a + 4\mathbb{Z}$. The negation is: there exists an integer n such that every integer m greater than n fails P . Equivalently $\exists n \forall m (m > n \Rightarrow \neg P(m))$. It does not assert that P fails for every integer, nor that m is at most n for every m .

Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.