

Exercise walk-through

17.1

Multivariable extrema and constrained optimisation

GRE · GRE Subject Test · GRE Mathematics

8. Choose and justify a method

Choose and justify a method

On a smooth equality constraint $g(x,y)=c$ with nonzero gradient g , solve gradient $f=\lambda$ gradient g together with $g=c$. These equations generate candidates; they do not classify or guarantee a global extremum. If the constraint gradient vanishes, the regularity condition fails and the multiplier equations can miss a constrained extremum. Treat such points separately. For $x^2+y^2=a$ and $xy=b>0$, $(x-y)^2=a-2b$ shows necessity $a\geq 2b$. It is also sufficient: take $s=\sqrt{a+2b}$, $d=\sqrt{a-2b}$, $x=(s+d)/2$ and $y=(s-d)/2$. These real values have $xy=b$ and $x^2+y^2=a$.

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A continuous function on a compact feasible set attains global extrema. To find them, compare all interior candidates and all boundary pieces, including corners and endpoints. For a rectangle, optimise the restrictions on each edge. For a disk, the circular boundary may use a parameter or multiplier. Solving only the unconstrained gradient ignores possible boundary winners.

17. Reasoning and check

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The circle is compact and its constraint gradient $(2x, 2y)$ never vanishes there.

Multipliers give $y = 2\lambda x$, $x = 2\lambda y$. Neither coordinate can be zero at a stationary point; hence $y = \pm x$. The four candidates $(1, 1)$, $(-1, -1)$, $(1, -1)$, $(-1, 1)$ give global maximum 1 and minimum -1. The bound $2|xy| \leq x^2 + y^2 = 2$ checks completeness.

On the line, $x^2 + (4 - x)^2 = 2(x - 2)^2 + 8$, so the minimum is 8 at $(2, 2)$. There is no maximum because the expression is unbounded. Multiplier candidates alone do not establish compactness.

Exercise 2.1

2.1 At a critical point $f_{xx}=2$, $f_{yy}=-2$ and $f_{xy}=0$. What does the second-derivative test give? [1]

- **A** Strict minimum
- **B** Strict maximum
- **C** Saddle point
- **D** Inconclusive
- **E** No critical point

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2.1 C. $D=2(-2)-0^2=-4$, so the point is a saddle. Minimum and maximum conflict with the negative Hessian determinant; inconclusive applies when that determinant is zero. The premise explicitly supplies a critical point.

Exercises 2.2·2.3

2.2 What is the maximum of xy subject to $x^2+y^2=2$?

[1]

- **A** -2
- **B** -1
- **C** 0
- **D** 1
- **E** 2

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2.2 D. $(x-y)^2 \geq 0$ implies $2xy \leq x^2+y^2=2$. Equality occurs at $(1,1)$ and $(-1,-1)$. Negative two and negative one cannot be the maximum because $(1,1)$ gives one. Zero misses positive values; two violates $2xy \leq 2$.

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2.3 For f the Hessian is $\text{diag}(2, -2)$ with determinant -4 ; the origin is a saddle. Along the axes the values have opposite signs. For g the Hessian at zero is zero, so the second-derivative test is inconclusive. Nevertheless $x^4 + y^4 \geq 0$, with equality only at the origin; it is a strict global minimum. An inconclusive test is not a classification of the point.

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Local four-mark guidance for each written task: choose and justify the method; carry through the reasoning; give every requested result; check the relevant restriction, hypothesis or boundary case. These are classroom marks, not GRE scoring.